

A bifurcation result for non-autonomous rational difference equations

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Autonomous:

$$x_n = \frac{\beta x_{n-1}}{1 + x_{n-2}}, \quad n = 1, 2, \dots$$

Here β is assumed to be a non-negative real number.

Non-autonomous:

$$x_n = \frac{\beta_n x_{n-1}}{1 + x_{n-2}}, \quad n = 1, 2, \dots$$

Here $\{\beta_n\}_{n=1}^{\infty}$ is assumed to be a non-negative sequence.

We assume that the parameter sequence is periodic.

When does periodicity preserve the qualitative behavior?

When does periodicity destroy the qualitative behavior?

When does periodicity preserve boundedness character?

When does periodicity destroy boundedness character?
(Camouzis, Ladas, 2006).

A Pattern of Boundedness:

$$x_n = \frac{Ax_{n-2}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$B_j = 0$ for $j = 2, 4, 6, 8, \dots \implies P_2$ Trichotomy.

$B_j > 0$ for some $j = 2, 4, 6, 8, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{Ax_{n-3}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$B_j = 0$ for $j = 3, 6, 9, 12, \dots \implies P_3$ Trichotomy.

$B_j > 0$ for some $j = 3, 6, 9, 12, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{Ax_{n-5}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$B_j = 0$ for $j = 5, 10, 15, 20, \dots \implies P_5$ Trichotomy.

$B_j > 0$ for some $j = 5, 10, 15, 20, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{Ax_{n-k}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$B_j = 0$ for $j = k, 2k, 3k, 4k, \dots \implies P_k$ Trichotomy.

$B_j > 0$ for some $j = k, 2k, 3k, 4k, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-2}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 3.

$B_j = 0$ for $j = 2, 4, 6, 8, \dots \implies P_6$ Trichotomy.

$B_j > 0$ for some $j = 2, 4, 6, 8, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-2}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 5.

$B_j = 0$ for $j = 2, 4, 6, 8, \dots \implies P_{10}$ Trichotomy.

$B_j > 0$ for some $j = 2, 4, 6, 8, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-2}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 4.

$B_j = 0$ for $j = 2, 4, 6, 8, \dots \implies P_4$ Trichotomy.

$B_j > 0$ for some $j = 2, 4, 6, 8, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-2}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 6.

$B_j = 0$ for $j = 2, 4, 6, 8, \dots \implies P_6$ Trichotomy.

$B_j > 0$ for some $j = 2, 4, 6, 8, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-3}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 6.

$B_j = 0$ for $j = 3, 6, 9, 12, \dots \implies P_6$ Trichotomy.

$B_j > 0$ for some $j = 3, 6, 9, 12, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-3}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 5.

$B_j = 0$ for $j = 3, 6, 9, 12, \dots \implies P_{15}$ Trichotomy.

$B_j > 0$ for some $j = 3, 6, 9, 12, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-3}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 2.

$B_j = 0$ for $j = 3, 6, 9, 12, \dots \implies P_6$ Trichotomy.

$B_j > 0$ for some $j = 3, 6, 9, 12, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-6}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 9.

$B_j = 0$ for $j = 6, 12, 18, 24, \dots \implies P_{18}$ Trichotomy.

$B_j > 0$ for some $j = 6, 12, 18, 24, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-10}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 15.

$B_j = 0$ for $j = 10, 20, 30, 40, \dots \implies P_{30}$ Trichotomy.

$B_j > 0$ for some $j = 10, 20, 30, 40, \dots \implies$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-k}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period p .

$B_j = 0$ for $j = k, 2k, 3k, 4k, \dots \implies$ Period $\frac{pk}{\gcd(p,k)}$ Trichotomy.

$B_j > 0$ for some $j = k, 2k, 3k, 4k, \dots \implies$ Bounded by Iteration.

Example:

$$x_n = \frac{A_n x_{n-3}}{1 + x_{n-1} + x_{n-2} + x_{n-4} + x_{n-5}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is a periodic with prime period 6.

Define:

$$C_0 = A_3 A_6$$

$$C_1 = A_1 A_4$$

$$C_2 = A_2 A_5$$

$$\max_{i=0,\dots,2} (C_i) < 1 \quad \Longrightarrow \quad 0 \text{ is GAS.}$$

$$\max_{i=0,\dots,2} (C_i) = 1 \quad \Longrightarrow \quad ESCP_6.$$

$$\max_{i=0,\dots,2} (C_i) > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{A_n x_{n-3}}{1 + x_{n-1} + x_{n-2} + x_{n-4} + x_{n-5}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is a periodic with prime period 4.

Define:

$$A_1 A_2 A_3 A_4 < 1 \quad \Longrightarrow \quad 0 \text{ is GAS.}$$

$$A_1 A_2 A_3 A_4 = 1 \quad \Longrightarrow \quad ESCP_{12}.$$

$$A_1 A_2 A_3 A_4 > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{A_n x_{n-5}}{1 + x_{n-1} + x_{n-3} + x_{n-6} + x_{n-9}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is a periodic with prime period 10.

Define:

$$C_0 = A_5 A_{10}$$

$$C_1 = A_1 A_6$$

$$C_2 = A_2 A_7$$

$$C_3 = A_3 A_8$$

$$C_4 = A_4 A_9$$

$$\max_{i=0,\dots,4} (C_i) < 1 \quad \Longrightarrow \quad 0 \text{ is GAS.}$$

$$\max_{i=0,\dots,4} (C_i) = 1 \quad \Longrightarrow \quad ESCP_{10}.$$

$$\max_{i=0,\dots,4} (C_i) > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{A_n x_{n-5}}{1 + x_{n-1} + x_{n-3} + x_{n-6} + x_{n-9}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is a periodic with prime period 7.

$$A_1 A_2 A_3 A_4 A_5 A_6 A_7 < 1 \quad \Longrightarrow \quad 0 \text{ is GAS.}$$

$$A_1 A_2 A_3 A_4 A_5 A_6 A_7 = 1 \quad \Longrightarrow \quad ESCP_{35}.$$

$$A_1 A_2 A_3 A_4 A_5 A_6 A_7 > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{A_n x_{n-6}}{1 + x_{n-1} + x_{n-3} + x_{n-9}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is a periodic with prime period 9.

Define:

$$C_0 = A_3 A_6 A_9$$

$$C_1 = A_1 A_4 A_7$$

$$C_2 = A_2 A_5 A_8$$

$$\max_{i=0,\dots,2} (C_i) < 1 \quad \Longrightarrow \quad 0 \text{ is GAS.}$$

$$\max_{i=0,\dots,2} (C_i) = 1 \quad \Longrightarrow \quad ESCP_{18}.$$

$$\max_{i=0,\dots,2} (C_i) > 1 \quad \Longrightarrow \quad \exists US.$$

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Theorem 1.

Consider the non-autonomous rational difference equation

$$x_n = \frac{A_n x_{n-k}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N}, \quad (1)$$

where $B_j \geq 0$ for all $j \in \{1, \dots, \ell\}$ and $\{A_n\}_{n=1}^{\infty}$ is a periodic sequence of prime period p with $A_n \geq 0$ for all $n \in \mathbb{N}$ and with non-negative initial conditions. Assume that $B_j = 0$ for all $j \equiv 0 \pmod{k}$. Define $r_m \in \mathbb{N}$ for $m \in \{0, \dots, k-1\}$ to be the smallest $r \leq \frac{p}{\gcd(p,k)}$ so that

$$\prod_{i=0}^{r-1} A_{n-ik} = \text{constant},$$

for all $n \equiv m \pmod{k}$. Define

$$C_m = \prod_{i=0}^{\frac{p}{\gcd(p,k)}-1} A_{pk+m-ik}$$

for $m \in \{0, \dots, k-1\}$.

Then solutions of Equation (1) exhibit the following behavior.

- i. When $C_m < 1$ for all $m \in \{0, \dots, k - 1\}$ then the zero equilibrium is globally asymptotically stable.
- ii. When $C_m \leq 1$ for all $m \in \{0, \dots, k - 1\}$ and also $C_m = 1$ for some $m \in \{0, \dots, k - 1\}$ then every solution converges to a periodic solution of not necessarily prime period $\frac{kp}{\gcd(p,k)}$. Moreover, let $I_1 = \{m \in \{0, \dots, k - 1\} | C_m = 1\}$. There exists periodic solutions of prime period kr_m for all $m \in I_1$.
- iii. When $C_m > 1$ for some $m \in \{0, \dots, k - 1\}$ then unbounded solutions exist for some choice of initial conditions.

Example:

Consider the non-autonomous rational difference equation

$$x_n = \frac{A_n x_{n-5}}{1 + x_{n-1} + x_{n-3} + x_{n-6} + x_{n-9}}, \quad n \in \mathbb{N}, \quad (2)$$

where $\{A_n\}_{n=1}^{\infty}$ is a periodic sequence of prime period 10 with $A_n \geq 0$ for all $n \in \mathbb{N}$ and with non-negative initial conditions. Define $r_m \in \mathbb{N}$ for $m \in \{0, \dots, 4\}$ to be the smallest $r \leq 2$ so that

$$\prod_{i=0}^{r-1} A_{n-5i} = \text{constant},$$

for all $n \equiv m \pmod{5}$. Define

$$C_m = \prod_{i=0}^1 A_{50+m-5i} = A_{m+5} A_{m+10}$$

for $m \in \{0, \dots, 4\}$.

So,

$$C_0 = A_5 A_{10}$$

$$C_1 = A_1 A_6$$

$$C_2 = A_2 A_7$$

$$C_3 = A_3 A_8$$

$$C_4 = A_4 A_9$$

Then solutions of Equation (2) exhibit the following behavior.

- i. When $C_m < 1$ for all $m \in \{0, \dots, 4\}$ then the zero equilibrium is globally asymptotically stable.
- ii. When $C_m \leq 1$ for all $m \in \{0, \dots, 4\}$ and also $C_m = 1$ for some $m \in \{0, \dots, 4\}$ then every solution converges to a periodic solution of not necessarily prime period 10. Moreover, let $I_1 = \{m \in \{0, \dots, 4\} | C_m = 1\}$. There exists periodic solutions of prime period kr_m for all $m \in I_1$.
- iii. When $C_m > 1$ for some $m \in \{0, \dots, 4\}$ then unbounded solutions exist for some choice of initial conditions.

A question:

Why do we need to consider the constants r_m ? Why can't part (ii) of the theorem be, "When $C_m \leq 1$ for all $m \in \{0, \dots, 4\}$ and also $C_m = 1$ for some $m \in \{0, \dots, 4\}$ then every solution converges to a periodic solution of not necessarily prime period 10. Moreover, there exists periodic solutions of prime period 10."?

Example:

Let $A_1 = 1$, $A_2 = 2$, $A_3 = 3$, $A_4 = 4$, $A_5 = 5$, $A_6 = 1$, $A_7 = 0$, $A_8 = 0$, $A_9 = 0$, and $A_{10} = 0$. Then every solution is eventually periodic with period 5.

A question:

How do we know that $r_m \leq 2$?

Answer:

Because of the forthcoming Lemma 1.

Lemma 1.

Let $\{A_n\}_{n=1}^{\infty}$ be a periodic sequence of prime period $p > 1$ with $A_n \geq 0$ for all $n \in \mathbb{N}$ and let

$$C_n = \prod_{i=0}^{\frac{p}{\gcd(p,k)}-1} A_{n-ik}.$$

Whenever $n \equiv m \pmod{k}$ then $C_n = C_m$.

Proof. Given n and m fixed with $n \equiv m \pmod{k}$, for each $i \in \{0, \dots, \frac{p}{\gcd(p,k)} - 1\}$ define $\sigma(i) = (\frac{m-n}{k} + i) \pmod{\frac{p}{\gcd(p,k)}}$. Notice that σ is a permutation on $\{0, \dots, \frac{p}{\gcd(p,k)} - 1\}$ so that $n - ik \equiv m - \sigma(i)k \pmod{p}$, and therefore $A_{n-ik} = A_{m-\sigma(i)k}$. This implies that all the terms of the product C_n are the same as the terms of the product C_m , possibly with reordering. Thus $C_n = C_m$.

Sketch of the Proof for Theorem 1.

Iterate the numerator $\frac{p}{\gcd(p,k)}$ times to obtain the difference inequality,

$$x_n \leq \prod_{i=0}^{\frac{p}{\gcd(p,k)}-1} A_{n-ik} x_{n-\frac{kp}{\gcd(p,k)}} = C_n \pmod{k} x_{n-\frac{kp}{\gcd(p,k)}}, \quad n \geq \frac{kp}{\gcd(p,k)}.$$

So if C_m is less than 1 for all $m \in \{0, \dots, k-1\}$ then every subsequence $\left\{ x_{\frac{nkp}{\gcd(p,k)}+d} \right\}_{n=0}^{\infty}$ must converge to 0 regardless of initial conditions. So every solution converges to 0, since all the subsequences do.

Sketch of the Proof for Theorem 1.

If C_m is less than or equal to 1 for all $m \in \{0, \dots, k-1\}$ then every subsequence $\left\{ x_{\frac{nkp}{\gcd(p,k)}+d} \right\}_{n=0}^{\infty}$ is monotone, and so must converge regardless of initial conditions. So every solution converges to a periodic solution of not necessarily prime period $\frac{kp}{\gcd(p,k)}$. To construct our unbounded solutions, if $C_m > 1$, then we choose initial conditions, $x_j = 0$ for $j \leq 0$ and $j \not\equiv m \pmod k$ and $x_j = 1$ for $j \leq 0$ and $j \equiv m \pmod k$.