

# **Local Stability and the m-M Theorem**

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The m-M theorem is a powerful global attractivity result introduced to the literature in (Kulenović, Ladas, and Sizer, 1998) and first applied to systems in (Kulenović, Ladas, and Overdeep, 2004).

Various generalizations of the m-M theorem have been used to prove global attractivity over the years. The most general version of the m-M theorem is given in (Kulenović and Merino, 2006).

At the University of Rhode Island, when trying to decide whether a difference equation, or system of difference equations has a globally attracting fixed point the m-M theorem is one of the first techniques we try to use.

Recently Gabriel Lugo and I were applying the m-M theorem to rational systems in the plane when we realized that the conclusion of global attractivity could be strengthened to global asymptotic stability without modifying the hypotheses of the theorem.

Today we will show how to generalize one version of the m-M theorem. The generalizations of the other versions follow similarly. We have a short note which is currently under review in the Journal of Difference Equations and Applications where we discuss this result more fully.

Now we state a version of the theorem along with the proof that appeared in (Kulenović and Ladas, 2002).

Let  $[a, b]$  be an interval of real numbers and assume that

$$f : [a, b] \times [a, b] \rightarrow [a, b]$$

is a continuous function satisfying the following properties:

(a)  $f(x, y)$  is non-increasing in  $x \in [a, b]$  for each  $y \in [a, b]$ , and  $f(x, y)$  is non-decreasing in  $y \in [a, b]$  for each  $x \in [a, b]$ ;

(b) The difference equation,

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, 2, \dots, \tag{1}$$

has no solutions of prime period two in  $[a, b]$ .

Then Equation 1 has a unique equilibrium  $\bar{x} \in [a, b]$  and  $\bar{x}$  is a global attractor in  $[a, b]$ .

*Proof.* We set  $m_0 = a$  and  $M_0 = b$  and for  $i \in \mathbb{N}$  we set  $M_i = f(m_{i-1}, M_{i-1})$  and  $m_i = f(M_{i-1}, m_{i-1})$ . Now observe that for each  $i \geq 0$

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0,$$

and

$$m_i \leq x_k \leq M_i \text{ for } k \geq 2i + 1. \quad (2)$$

Now set  $m = \lim_{i \rightarrow \infty} m_i$  and  $M = \lim_{i \rightarrow \infty} M_i$  then by continuity we have  $m = f(M, m)$  and  $M = f(m, M)$ . Thus due to (b)  $m = M$ .

□

Now we state and prove a new version of the theorem where global attractivity is strengthened to global asymptotic stability.

Let  $[a, b]$  be an interval of real numbers and assume that

$$f : [a, b] \times [a, b] \rightarrow [a, b]$$

is a continuous function satisfying the following properties:

(a)  $f(x, y)$  is non-increasing in  $x \in [a, b]$  for each  $y \in [a, b]$ , and  $f(x, y)$  is non-decreasing in  $y \in [a, b]$  for each  $x \in [a, b]$ ;

(b) The difference equation,

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, 2, \dots, \tag{3}$$

has no solutions of prime period two in  $[a, b]$ .

Then Equation 3 has a unique equilibrium  $\bar{x} \in [a, b]$  and  $\bar{x}$  is globally asymptotically stable in  $[a, b]$ .

Proof:

The beginning of the proof is the same as before. We set  $m_0 = a$  and  $M_0 = b$  and for  $i \in \mathbb{N}$  we set  $M_i = f(m_{i-1}, M_{i-1})$  and  $m_i = f(M_{i-1}, m_{i-1})$ . Now observe that for each  $i \geq 0$

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0,$$

and

$$m_i \leq x_k \leq M_i \text{ for } k \geq 2i + 1. \quad (4)$$

Now set  $m = \lim_{i \rightarrow \infty} m_i$  and  $M = \lim_{i \rightarrow \infty} M_i$  then by continuity we have  $m = f(M, m)$  and  $M = f(m, M)$ . Thus due to (b)  $m = M$ .

For the remainder of the proof it helps to consider a map  $g : [a, b] \times [a, b] \rightarrow [a, b] \times [a, b]$  where  $g(x, y) = (f(x, y), x)$ . Notice that  $\lim_{i \rightarrow \infty} m_i = \bar{x} = \lim_{i \rightarrow \infty} M_i$ . Thus given  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  so that  $[m_N, M_N] \times [m_N, M_N]$  is contained in the epsilon ball  $B((\bar{x}, \bar{x}), \epsilon)$ .

We also know by the continuity of  $g$  that given  $\epsilon > 0$  there exists  $\delta > 0$  so that if  $y \in B((\bar{x}, \bar{x}), \delta)$  then  $g^n(y) \in B((\bar{x}, \bar{x}), \epsilon)$  for all  $n \leq 2N + 3$ .

In case the previous statement is unclear let me clarify. Given a fixed  $n$ , the continuity of  $g$  implies the continuity of  $g^n$ . So, for that fixed  $n$ , given  $\epsilon > 0$  there exists  $\delta_n > 0$  so that if  $y \in B((\bar{x}, \bar{x}), \delta_n)$  then  $g^n(y) \in B(g^n((\bar{x}, \bar{x})), \epsilon) = B((\bar{x}, \bar{x}), \epsilon)$ . If we take  $\delta = \min_{n \leq 2N+3}(\delta_n)$  then we have found the delta needed in the previous slide.

Thus given  $\epsilon > 0$  there exists  $\delta > 0$  so that if  $y \in B((\bar{x}, \bar{x}), \delta)$  then  $g^n(y) \in B((\bar{x}, \bar{x}), \epsilon)$  for all  $n \in \mathbb{N}$ .

Thus  $\bar{x}$  is locally stable and a global attractor in  $[a, b]$ . Thus  $\bar{x}$  is globally asymptotically stable in  $[a, b]$ .

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