# On the boundedness character of rational systems in the plane

Gabriel Lugo[*] $and Frank J. Palladino$
University of Rhode Island
Department of Mathematics
Kingston, RI 02881-0816, USA
frank@math.uri.edu
glugo@math.uri.edu

Today we will present a key theorem from the paper:

G. Lugo and F. Palladino, On the boundedness character of rational systems in the plane, *J. Difference Equ. Appl.* in press.

This theorem establishes the conjectures in (Ladas and Camouzis, 2010) regarding the systems labeled (23,23), (23,31), (23,34), (23,46), (31,31), (31,34), (31,46), (34,34), (34,46), and (46,46) with the numbering system from (Camouzis, Kulenović, Ladas, and Merino, 2009).

System (23,23)

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{B_1 x_n}, n = 0, 1, 2, \dots,$$

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{C_2 y_n}, n = 0, 1, 2, \dots.$$

System (23,31)

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{B_1 x_n}, n = 0, 1, 2, \dots,$$

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{A_2 + C_2 y_n}, n = 0, 1, 2, \dots.$$

System (23,34)

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{B_1 x_n}, n = 0, 1, 2, \dots,$$

$$y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + C_2 y_n}, n = 0, 1, 2, \dots.$$

System (23,46)

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{B_1 x_n}, n = 0, 1, 2, \dots,$$

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + C_2 y_n}, n = 0, 1, 2, \dots.$$

System (31,31)

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{A_1 + B_1 x_n}, n = 0, 1, 2, \dots,$$

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{A_2 + C_2 y_n}, n = 0, 1, 2, \dots.$$

System (31,34)

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{A_1 + B_1 x_n}, n = 0, 1, 2, \dots,$$

$$y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + C_2 y_n}, n = 0, 1, 2, \dots.$$

System (31,46)

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{A_1 + B_1 x_n}, n = 0, 1, 2, \dots,$$

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + C_2 y_n}, n = 0, 1, 2, \dots.$$

System (34,34)

$$x_{n+1}=\frac{\beta_1 x_n+\gamma_1 y_n}{A_1+B_1 x_n}, n=0,1,2,\ldots,$$

$$y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{A_2+C_2 y_n}, n=0,1,2,\ldots.$$

System (34,46)

$$x_{n+1}=\frac{\beta_1 x_n+\gamma_1 y_n}{A_1+B_1 x_n}, n=0,1,2,\ldots,$$

$$y_{n+1}=\frac{\alpha_2+\beta_2 x_n+\gamma_2 y_n}{A_2+C_2 y_n}, n=0,1,2,\ldots.$$

System (46,46)

$$x_{n+1}=\frac{\alpha_1+\beta_1 x_n+\gamma_1 y_n}{A_1+B_1 x_n}, n=0,1,2,\ldots,$$

$$y_{n+1}=\frac{\alpha_2+\beta_2 x_n+\gamma_2 y_n}{A_2+C_2 y_n}, n=0,1,2,\ldots.$$

Theorem:

Consider the following first order system of two rational difference equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n}, n = 0, 1, 2, \dots,$$

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + C_2 y_n}, n = 0, 1, 2, \dots.$$

We assume non-negative parameters and non-negative initial conditions. We further assume the following

1. $B_1, C_2, \beta_2, \gamma_1 > 0$,
2. $\frac{\gamma_1}{A_1+B_1} > 2$ and $\frac{\beta_2}{A_2+C_2} > 1$,
3. $\frac{\alpha_1+1+\beta_1}{B_1} < 1$ and $\frac{\alpha_2+1+\gamma_2}{C_2} < 1$.

then the solutions $x_n$ and $y_n$ are unbounded for some non-negative initial conditions.

Proof:

We first prove by induction that under certain non-negative initial conditions $y_{2n} > max(1, \gamma_1, \beta_2)$ and $x_{2n} < 1$. We choose the initial conditions to provide the base case. Let $y_0 > max(1, \gamma_1, \beta_2)$ and $x_0 < 1$. Now let us prove the inductive step. Assume $y_{2n-2} > max(1, \gamma_1, \beta_2)$ and $x_{2n-2} < 1$, then we have

$$x_{2n-1} = \frac{\alpha_1 + \beta_1 x_{2n-2} + \gamma_1 y_{2n-2}}{A_1 + B_1 x_{2n-2}} \geq \frac{\gamma_1 y_{2n-2}}{A_1 + B_1 x_{2n-2}} > \frac{\gamma_1 y_{2n-2}}{A_1 + B_1}.$$

We have assumed that $\frac{\gamma_1}{A_1+B_1} > 2$ so we have that $x_{2n-1} > 2y_{2n-2}$.

Furthermore we have the following

$$y_{2n-1} = \frac{\alpha_2 + \beta_2 x_{2n-2} + \gamma_2 y_{2n-2}}{A_2 + C_2 y_{2n-2}} \leq \frac{\alpha_2 + \beta_2 x_{2n-2} + \gamma_2 y_{2n-2}}{C_2 y_{2n-2}}$$

$$< \frac{\alpha_2 y_{2n-2} + y_{2n-2} + \gamma_2 y_{2n-2}}{C_2 y_{2n-2}} = \frac{\alpha_2 + 1 + \gamma_2}{C_2} < 1.$$

Thus $y_{2n-1} < 1$. Now we use these facts to get the following

$$y_{2n} = \frac{\alpha_2 + \beta_2 x_{2n-1} + \gamma_2 y_{2n-1}}{A_2 + C_2 y_{2n-1}} \geq \frac{\beta_2 x_{2n-1}}{A_2 + C_2 y_{2n-1}} > \frac{\beta_2 x_{2n-1}}{A_2 + C_2}.$$

We have assumed that $\frac{\beta_2}{A_2+C_2} > 1$ so we have that $y_{2n} > x_{2n-1} > 2y_{2n-2} > max(1, \gamma_1, \beta_2)$.

Also, since $x_{2n-1} > 2y_{2n-2} > max(1, \gamma_1, \beta_2)$ ,we have the following

$$x_{2n} = \frac{\alpha_1 + \beta_1 x_{2n-1} + \gamma_1 y_{2n-1}}{A_1 + B_1 x_{2n-1}} \leq \frac{\alpha_1 x_{2n-1} + \beta_1 x_{2n-1} + x_{2n-1}}{B_1 x_{2n-1}} = \frac{\alpha_1 + \beta_1 + 1}{B_1}.$$

We have assumed that $\frac{\alpha_1+1+\beta_1}{B_1} < 1$ so we have that $x_{2n} < 1$.

Thus we have shown that $y_{2n} > max(1, \gamma_1, \beta_2)$ and $x_{2n} < 1$ for all $n \in \mathbb{N}$. Notice that we have already shown that this implies that $y_{2n} > x_{2n-1} > 2y_{2n-2}$ for all $n \in \mathbb{N}$ hence $lim_{n \to \infty} y_{2n} = \infty$ and $lim_{n \to \infty} x_{2n+1} = \infty$.

## References

**1.** E. Camouzis, M.R.S. Kulenović, G. Ladas, and O. Merino, Rational systems in the plane, *J. Difference Equa. Appl.* **15**(2009), 303-323.

**2.** E. Camouzis and G. Ladas, Global results on rational systems in the plane, Part 1, *J. Difference Equa. Appl.* **15**(2009), in press.

**3.** F.J. Palladino, Difference inequalities, comparison tests, and some consequences, *Involve* **1**(2008), 91-100.