A bifurcation result for non-autonomous rational difference equations

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The Beverton-Holt model:

Originally introduced by Raymond J.H. Beverton and Sidney J. Holt in 1957 to study the dynamics of exploited fish populations.

Originally developed to study population dynamics in single species industrialized fisheries.

The Beverton-Holt model:

$$x_n = \frac{Rx_{n-1}}{1 + x_{n-1}}, \qquad n = 1, 2, \ldots .$$

The parameter R is interpreted as the proliferation rate.

When $R > 1$ the carrying capacity is the positive equilibrium $R - 1$.

When $R \leq 1$ the 0 equilibrium is globally asymptotically stable.

The Beverton-Holt model can be solved explicitly.

Make the change of variables $y_n = \frac{1}{x_n}$.

The Beverton-Holt model becomes:

$$\frac{1}{y_n} = \frac{\frac{R}{y_{n-1}}}{1 + \frac{1}{y_{n-1}}}, \quad n = 1, 2, \ldots .$$

So,

$$\frac{1}{y_n} = \frac{R}{y_{n-1} + 1}, \quad n = 1, 2, \ldots .$$

So,

$$y_n = \frac{y_{n-1} + 1}{R}, \quad n = 1, 2, \ldots .$$

So if $R = 1$ and $x_0 \neq 0$, then

$$y_n = y_0 + n, \quad n = 1, 2, \ldots.$$

So,

$$x_n = \frac{x_0}{1 + n x_0}, \quad n = 1, 2, \ldots.$$

If $R \neq 1$ and $x_0 \neq 0$, then

$$y_n = \left(\frac{1}{R}\right)^n \left(y_0 - \frac{1}{R-1}\right) + \frac{1}{R-1}, \quad n = 1, 2, \ldots.$$

So,

$$x_n = \frac{R^n(R-1)x_0}{R-1+R^n x_0 - x_0}, \quad n = 1, 2, \ldots.$$

Periodic forcing is used to model seasonal changes.

The periodically forced Beverton-Holt model:

$$x_n = \frac{R_n x_{n-1}}{1 + x_{n-1}}, \quad n = 1, 2, \ldots .$$

Where the positive sequence $\{R_n\}_{n=0}^{\infty}$ is periodic with period $P > 1$.

The periodically forced Beverton-Holt model can also be solved explicitly.

Make the change of variables $y_n = \frac{1}{x_n}$.

We get,

$$y_n = \frac{y_{n-1} + 1}{R_n}, \quad n = 1, 2, \ldots .$$

We iterating we get,

$$y_n = \frac{y_{n-2} + 1}{R_n R_{n-1}} + \frac{1}{R_n}, \quad n = 2, 3, \ldots .$$

$$\vdots$$

$$y_n = \sum_{i=1}^{P} \left(\frac{1}{\prod_{j=1}^{i} R_{n+1-j}} \right) + \frac{y_{n-P}}{\prod_{j=1}^{P} R_{n+1-j}}, \quad n = P, P+1, \ldots .$$

Call

$$a_n = \sum_{i=1}^{P} \left(\frac{1}{\prod_{j=1}^{i} R_{n+1-j}} \right) \quad and \quad b_n = \frac{1}{\prod_{j=1}^{P} R_{n+1-j}}.$$

Since $\{R_n\}_{n=0}^{\infty}$ is periodic with not necessarily prime period $P > 1$ so are $\{a_n\}_{n=P+1}^{\infty}$ and $\{b_n\}_{n=P+1}^{\infty}$.

In fact b_n is constant and we will call it B from now on.

So,

$$y_{Pn+\ell} = a_{P+\ell} + By_{P(n-1)+\ell}, \quad n = P, P+1, \ldots .$$

Since $a_{Pn+\ell} = a_{P+\ell}$ for all $n \in \mathbb{N}$.

So we have decoupled the solution into subsequences which we can solve.

The explicit solution can now be obtained with some tedious computation.

Pielou’s equation:

Originally introduced by Evelyn C. Pielou.

The equation has been used to model the dynamics of populations of blow-flies, (Lucilia Cuprind).

Pielou's equation:

$$x_n = \frac{Rx_{n-1}}{1 + x_{n-2}}, \qquad n = 1, 2, \ldots .$$

The parameter R is interpreted as the proliferation rate.

When $R > 1$ the carrying capacity is the positive equilibrium $R - 1$.

When $R \leq 1$ the 0 equilibrium is globally asymptotically stable.

Pielou's equation:

$$x_n = \frac{Rx_{n-1}}{1 + x_{n-2}}, \quad n = 1, 2, \ldots .$$

When $R > 1$ the positive equilibrium $R - 1$ is a global attractor for all positive solutions.

When $R \leq 1$ the 0 equilibrium is globally asymptotically stable.

Proof:

Suppose $R \leq 1$, then

$$x_n = \frac{Rx_{n-1}}{1+x_{n-2}} \leq Rx_{n-1}, \quad n = 1, 2, \ldots .$$

So, the solution is monotone decreasing and bounded below by zero, and thus must converge to an equilibrium.

The 0 equilibrium is the only equilibrium in this case, thus the zero equilibrium is globally asymptotically stable.

Proof:

Suppose $R > 1$, then

$$x_n = \frac{Rx_{n-1}}{1 + x_{n-2}} = \frac{R^2 x_{n-2}}{(1 + x_{n-2})(1 + x_{n-3})} \leq R^2, \quad n = 2, 3, \ldots.$$

So, the solution is bounded above by R^2.

We claim that every positive solution is also bounded below by a positive constant.

Suppose not then we may take a subsequence $\{x_{n_i}\}$ so that

$$x_{n_i} \rightarrow 0$$

and so that for each $i \in \mathbb{N}$

$$x_{n_i} < x_j \;\; for \; all \;\; j < n_i.$$

Proof:

However then $\{x_{n_i}\}$, $\{x_{n_i-1}\}$, and $\{x_{n_i-1}\}$ all converge to 0.

So for i large enough $x_{n_i-2} < R-1$,

$$x_{n_i} = \frac{Rx_{n_i-1}}{1+x_{n_i-2}} > \frac{Rx_{n_i-1}}{1+R-1} = x_{n_i-1},$$

a contradiction.

Proof:

Now take $S = limsup_{n\to\infty} x_n$ and $I = liminf_{n\to\infty} x_n$.

Since

$$x_n = \frac{R^2 x_{n-2}}{(1+x_{n-2})(1+x_{n-3})}, \quad n = 2, 3, \ldots .$$

We get

$$S \leq \frac{R^2 S}{(1+S)(1+I)},$$

and

$$I \geq \frac{R^2 I}{(1+I)(1+S)}.$$

Proof:

So,

$$(1+S)(1+I) = R^2.$$

Take a subsequence $\{x_{n_j}\}$ so that $x_{n_j} \to S$, $x_{n_j-1} \to L_1$, $x_{n_j-2} \to L_2$, $x_{n_j-3} \to L_3$, $x_{n_j-4} \to L_4$, and $x_{n_j-5} \to L_5$.

Proof:

Then,

$$S = \frac{R^2 L_2}{(1 + L_2)(1 + L_3)}.$$

This forces $L_2 = S$ and $L_3 = I$. Since if not then

$$S = \frac{R^2 L_2}{(1 + L_2)(1 + L_3)} < \frac{R^2 S}{(1 + S)(1 + I)} = S.$$

Proof:

So, $L_2 = S$ and $L_3 = I$. Similarly, $L_2 = S$ forces $L_4 = S$ and $L_5 = I$

So,

$$S \geq L_3 = \frac{RS}{1+I} \geq S.$$

So $S = I$.

Periodically forced Pielou's equation:

$$x_n = \frac{R_n x_{n-1}}{1 + x_{n-2}}, \quad n = 1, 2, \ldots .$$

Where $\{R_n\}$ is periodic with period $P > 1$.

In this case if $R_1 \cdots R_p > 1$ then every solution converges to a periodic solution of not necessarily prime period P.

If $R_1 \cdots R_p \leq 1$ then every solution converges to the zero equilibrium.

The proof follows along similar lines to the previous proof but is too long to show.

See

E. Camouzis and G. Ladas, Periodically Forced Pielou's equation, *J.Math Anal. Appl.* **333**(2007), 117-127.

How about the equation:

$$x_n = \frac{4x_{n-1}}{1 + x_{n-3}}, \quad n = 1, 2, \ldots .$$

Conjecture: Almost all solutions exhibit conservative chaos.

When does periodicity preserve the qualitative behavior?

When does periodicity destroy the qualitative behavior?

When does periodicity preserve boundedness character?

When does periodicity destroy boundedness character?
(Camouzis, Ladas, 2006).

A Pattern of Boundedness:

$$x_n = \frac{Ax_{n-2}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$B_j = 0$ for $j = 2, 4, 6, 8, \ldots \quad \Longrightarrow \quad P_2$ Trichotomy.

$B_j > 0$ for some $j = 2, 4, 6, 8, \ldots \quad \Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A x_{n-3}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$B_j = 0$ for $j = 3, 6, 9, 12, \ldots \quad \Longrightarrow \quad P_3$ Trichotomy.

$B_j > 0$ for some $j = 3, 6, 9, 12, \ldots \quad \Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{Ax_{n-5}}{1+\sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$B_j = 0$ for $j = 5, 10, 15, 20, \ldots$ $\Longrightarrow$ P_5 Trichotomy.

$B_j > 0$ for some $j = 5, 10, 15, 20, \ldots$ $\Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A x_{n-k}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$B_j = 0$ for $j = k, 2k, 3k, 4k, \ldots \quad \Longrightarrow \quad P_k$ Trichotomy.

$B_j > 0$ for some $j = k, 2k, 3k, 4k, \ldots \quad \Longrightarrow \quad$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-2}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 3.

$B_j = 0$ for $j = 2, 4, 6, 8, \ldots \quad \Longrightarrow \quad P_6$ Trichotomy.

$B_j > 0$ for some $j = 2, 4, 6, 8, \ldots \quad \Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-2}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 5.

$B_j = 0$ for $j = 2, 4, 6, 8, \ldots \quad \Longrightarrow \quad P_{10}$ Trichotomy.

$B_j > 0$ for some $j = 2, 4, 6, 8, \ldots \quad \Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-2}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 4.

$B_j = 0$ for $j = 2, 4, 6, 8, \ldots \quad \Longrightarrow \quad P_4$ Trichotomy.

$B_j > 0$ for some $j = 2, 4, 6, 8, \ldots \quad \Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-2}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 6.

$B_j = 0$ for $j = 2, 4, 6, 8, \ldots \quad \Longrightarrow \quad P_6$ Trichotomy.

$B_j > 0$ for some $j = 2, 4, 6, 8, \ldots \quad \Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-3}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 6.

$B_j = 0$ for $j = 3, 6, 9, 12, \ldots \quad \Longrightarrow \quad P_6$ Trichotomy.

$B_j > 0$ for some $j = 3, 6, 9, 12, \ldots \quad \Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-3}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 5.

$B_j = 0$ for $j = 3, 6, 9, 12, \ldots \quad \Longrightarrow \quad P_{15}$ Trichotomy.

$B_j > 0$ for some $j = 3, 6, 9, 12, \ldots \quad \Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-3}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 2.

$B_j = 0$ for $j = 3, 6, 9, 12, \ldots \quad \Longrightarrow \quad P_6$ Trichotomy.

$B_j > 0$ for some $j = 3, 6, 9, 12, \ldots \quad \Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-6}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 9.

$B_j = 0$ for $j = 6, 12, 18, 24, \ldots \quad \Longrightarrow \quad P_{18}$ Trichotomy.

$B_j > 0$ for some $j = 6, 12, 18, 24, \ldots \quad \Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-10}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period 15.

$B_j = 0$ for $j = 10, 20, 30, 40, \ldots \quad \Longrightarrow \quad P_{30}$ Trichotomy.

$B_j > 0$ for some $j = 10, 20, 30, 40, \ldots \quad \Longrightarrow$ Bounded by Iteration.

A Pattern of Boundedness:

$$x_n = \frac{A_n x_{n-k}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is periodic with prime period p.

$B_j = 0$ for $j = k, 2k, 3k, 4k, \ldots$ $\Longrightarrow$ Period $\frac{pk}{gcd(p,k)}$ Trichotomy.

$B_j > 0$ for some $j = k, 2k, 3k, 4k, \ldots$ $\Longrightarrow$ Bounded by Iteration.

Example:

$$x_n = \frac{A_n x_{n-3}}{1 + x_{n-1} + x_{n-2} + x_{n-4} + x_{n-5}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is a periodic with prime period 6.
Define:

$$C_0 = A_3 A_6$$
$$C_1 = A_1 A_4$$
$$C_2 = A_2 A_5$$

$$\max_{i=0,\dots,2} (C_i) < 1 \quad \Longrightarrow \quad 0 \;\; is \;\; GAS.$$

$$\max_{i=0,\dots,2} (C_i) = 1 \quad \Longrightarrow \quad ESCP_6.$$

$$\max_{i=0,\dots,2} (C_i) > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{A_n x_{n-3}}{1 + x_{n-1} + x_{n-2} + x_{n-4} + x_{n-5}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is a periodic with prime period 4.

Define:

$$A_1A_2A_3A_4 < 1 \quad \Longrightarrow \quad 0 \;\; is \;\; GAS.$$

$$A_1A_2A_3A_4 = 1 \quad \Longrightarrow \quad ESCP_{12}.$$

$$A_1A_2A_3A_4 > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{A_n x_{n-5}}{1 + x_{n-1} + x_{n-3} + x_{n-6} + x_{n-9}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is a periodic with prime period 10.

Define:

$$C_0 = A_5 A_{10}$$
$$C_1 = A_1 A_6$$
$$C_2 = A_2 A_7$$
$$C_3 = A_3 A_8$$
$$C_4 = A_4 A_9$$

$$\max_{i=0,\ldots,4} (C_i) < 1 \quad \Longrightarrow \quad 0 \ \ is \ \ GAS.$$

$$\max_{i=0,\ldots,4} (C_i) = 1 \quad \Longrightarrow \quad ESCP_{10}.$$

$$\max_{i=0,\ldots,4} (C_i) > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{A_n x_{n-5}}{1 + x_{n-1} + x_{n-3} + x_{n-6} + x_{n-9}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is a periodic with prime period 7.

$$A_1A_2A_3A_4A_5A_6A_7 < 1 \quad \Longrightarrow \quad 0 \;\; is \;\; GAS.$$

$$A_1A_2A_3A_4A_5A_6A_7 = 1 \quad \Longrightarrow \quad ESCP_{35}.$$

$$A_1A_2A_3A_4A_5A_6A_7 > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{A_n x_{n-6}}{1 + x_{n-1} + x_{n-3} + x_{n-9}}, \quad n \in \mathbb{N},$$

$\{A_n\}_{n=1}^{\infty}$ is a periodic with prime period 9.

Define:

$$\begin{aligned} C_0 &= A_3 A_6 A_9 \\ C_1 &= A_1 A_4 A_7 \\ C_2 &= A_2 A_5 A_8 \end{aligned}$$

$$\max_{i=0,\dots,2}(C_i) < 1 \quad \Longrightarrow \quad 0 \;\; is \;\; GAS.$$

$$\max_{i=0,\dots,2}(C_i) = 1 \quad \Longrightarrow \quad ESCP_{18}.$$

$$\max_{i=0,\dots,2}(C_i) > 1 \quad \Longrightarrow \quad \exists US.$$

$$x_n = \frac{A_n x_{n-k}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

Assume $B_j = 0$ for $j = k, 2k, 3k, \ldots$.
$\{A_n\}_{n=1}^{\infty}$ is a periodic with prime period p.
Define:

$$C_m = \prod_{i=0}^{\frac{p}{gcd(p,k)}-1} A_{pk+m-ik}$$

for $m \in \{0, \ldots, k-1\}$.

$$\max_{i=0,\ldots,k-1} (C_i) < 1 \quad \Longrightarrow \quad 0 \ \ is \ \ GAS.$$

$$\max_{i=0,\ldots,k-1} (C_i) = 1 \quad \Longrightarrow \quad ESCP \frac{pk}{gcd(p,k)}.$$

$$\max_{i=0,\ldots,k-1} (C_i) > 1 \quad \Longrightarrow \quad \exists US.$$

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Theorem 1.

Consider the non-autonomous rational difference equation

$$x_n = \frac{A_n x_{n-k}}{1 + \sum_{j=1}^{\ell} B_j x_{n-j}}, \quad n \in \mathbb{N}, \tag{1}$$

where $B_j \geq 0$ for all $j \in \{1, \dots, \ell\}$ and $\{A_n\}_{n=1}^{\infty}$ is a periodic sequence of prime period p with $A_n \geq 0$ for all $n \in \mathbb{N}$ and with non-negative initial conditions. Assume that $B_j = 0$ for all $j \equiv 0 \mod k$. Define $r_m \in \mathbb{N}$ for $m \in \{0, \dots, k-1\}$ to be the smallest $r \leq \frac{p}{gcd(p,k)}$ so that

$$\prod_{i=0}^{r-1} A_{n-ik} = constant,$$

for all $n \equiv m \mod k$. Define

$$C_m = \prod_{i=0}^{\frac{p}{gcd(p,k)}-1} A_{pk+m-ik}$$

for $m \in \{0, \dots, k-1\}$.

Then solutions of Equation (1) exhibit the following behavior.

i. When $C_m < 1$ for all $m \in \{0, \ldots, k-1\}$ then the zero equilibrium is globally asymptotically stable.

ii. When $C_m \leq 1$ for all $m \in \{0, \ldots, k-1\}$ and also $C_m = 1$ for some $m \in \{0, \ldots, k-1\}$ then every solution converges to a periodic solution of not necessarily prime period $\frac{kp}{gcd(p,k)}$. Moreover, let $I_1 = \{m \in \{0, \ldots, k-1\} | C_m = 1\}$. There exists periodic solutions of prime period kr_m for all $m \in I_1$.

iii. When $C_m > 1$ for some $m \in \{0, \ldots, k-1\}$ then unbounded solutions exist for some choice of initial conditions.

Example:

Consider the non-autonomous rational difference equation

$$x_n = \frac{A_n x_{n-5}}{1 + x_{n-1} + x_{n-3} + x_{n-6} + x_{n-9}}, \quad n \in \mathbb{N}, \tag{2}$$

where $\{A_n\}_{n=1}^{\infty}$ is a periodic sequence of prime period 10 with $A_n \geq 0$ for all $n \in \mathbb{N}$ and with non-negative initial conditions. Define $r_m \in \mathbb{N}$ for $m \in \{0, \ldots, 4\}$ to be the smallest $r \leq 2$ so that

$$\prod_{i=0}^{r-1} A_{n-5i} = constant,$$

for all $n \equiv m \mod 5$. Define

$$C_m = \prod_{i=0}^{1} A_{50+m-5i} = A_{m+5} A_{m+10}$$

for $m \in \{0, \ldots, 4\}$.

So,

$$C_0 = A_5 A_{10}$$
$$C_1 = A_1 A_6$$
$$C_2 = A_2 A_7$$
$$C_3 = A_3 A_8$$
$$C_4 = A_4 A_9$$

Then solutions of Equation (2) exhibit the following behavior.

i. When $C_m < 1$ for all $m \in \{0, \ldots, 4\}$ then the zero equilibrium is globally asymptotically stable.

ii. When $C_m \leq 1$ for all $m \in \{0, \ldots, 4\}$ and also $C_m = 1$ for some $m \in \{0, \ldots, 4\}$ then every solution converges to a periodic solution of not necessarily prime period 10. Moreover, let $I_1 = \{m \in \{0, \ldots, 4\} | C_m = 1\}$. There exists periodic solutions of prime period kr_m for all $m \in I_1$.

iii. When $C_m > 1$ for some $m \in \{0, \ldots, 4\}$ then unbounded solutions exist for some choice of initial conditions.

A question:

Why do we need to consider the constants r_m? Why can't part (ii) of the theorem be, "When $C_m \leq 1$ for all $m \in \{0, \ldots, 4\}$ and also $C_m = 1$ for some $m \in \{0, \ldots, 4\}$ then every solution converges to a periodic solution of not necessarily prime period 10. Moreover, there exists periodic solutions of prime period 10."?

Example:

Let $A_1 = 1$, $A_2 = 2$, $A_3 = 3$, $A_4 = 4$, $A_5 = 5$, $A_6 = 1$, $A_7 = 0$, $A_8 = 0$, $A_9 = 0$, and $A_{10} = 0$. Then every solution is eventually periodic with period 5.

A question:
How do we know that $r_m \leq 2$?
Answer:
Because of the forthcoming Lemma 1.

Lemma 1.

Let $\{A_n\}_{n=1}^{\infty}$ be a periodic sequence of prime period $p > 1$ with $A_n \geq 0$ for all $n \in \mathbb{N}$ and let

$$C_n = \prod_{i=0}^{\frac{p}{gcd(p,k)}-1} A_{n-ik}.$$

Whenever $n \equiv m \mod k$ then $C_n = C_m$.

Proof. Given n and m fixed with $n \equiv m \mod k$, for each $i \in \{0, \ldots, \frac{p}{gcd(p,k)} - 1\}$ define $\sigma(i) = (\frac{m-n}{k} + i) \mod \frac{p}{gcd(p,k)}$. Notice that σ is a permutation on $\{0, \ldots, \frac{p}{gcd(p,k)} - 1\}$ so that $n - ik \equiv m - \sigma(i)k \mod p$, and therefore $A_{n-ik} = A_{m-\sigma(i)k}$. This implies that all the terms of the product C_n are the same as the terms of the product C_m, possibly with reordering. Thus $C_n = C_m$.

Sketch of the Proof for Theorem 1.

Iterate the numerator $\frac{p}{gcd(p,k)}$ times to obtain the difference inequality,

$$x_n \le \prod_{i=0}^{\frac{p}{gcd(p,k)}-1} A_{n-ik} x_{n-\frac{kp}{gcd(p,k)}} = C_{n \mod k} x_{n-\frac{kp}{gcd(p,k)}}, \quad n \ge \frac{kp}{gcd(p,k)}.$$

So if C_m is less than 1 for all $m \in \{0, \ldots, k-1\}$ then every subsequence $\left\{x_{\frac{nkp}{gcd(p,k)}+d}\right\}_{n=0}^{\infty}$ must converge to 0 regardless of initial conditions. So every solution converges to 0, since all the subsequences do.

Sketch of the Proof for Theorem 1.

If C_m is less than or equal to 1 for all $m \in \{0, \ldots, k-1\}$ then every subsequence $\left\{x_{\frac{nkp}{gcd(p,k)}+d}\right\}_{n=0}^{\infty}$ is monotone, and so must converge regardless of initial conditions. So every solution converges to a periodic solution of not necessarily prime period $\frac{kp}{gcd(p,k)}$.

To construct our unbounded solutions, if $C_m > 1$, then we choose initial conditions, $x_j = 0$ for $j \leq 0$ and $j \not\equiv m \mod k$ and $x_j = 1$ for $j \leq 0$ and $j \equiv m \mod k$.