UNBOUNDEDNESS RESULTS FOR SYSTEMS

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Abstract. We study k^{th} order systems of two rational difference equations

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, n \in \mathbb{N}.$$

In particular we assume non-negative parameters and non-negative initial conditions. We develop several approaches which allow us to prove that unbounded solutions exist for certain initial conditions in a range of the parameters.

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1. Introduction

There has been a recent interest in the study of systems of rational difference equations. Our goal is to provide several general theorems which prove the existence of unbounded solutions for systems of rational difference equations. It is important to realize that these theorems only apply in a range of the parameters and that certain assumptions are placed on the initial conditions in order to achieve unbounded solutions. We will proceed in the following manner. First we will introduce the reader to the source of the idea for the theorem. For example if the idea arose from the study of cerain special cases, we will present these cases and describe how they motivate the subsequent theorem. If the idea was adapted from prior results, which do not originally apply to systems, we will of course cite the result, and then describe in detail the adaptations necessary.

Before beginning let us look closely at our notation. We find that often times for rational difference equations the behavior can change in dramatic ways depending on whether a particular parameter is zero or positive. It is for this reason that we adopt a notation similar to that presented in theorem 6 of [3]. So we let $I_\beta = \{i \in \{1,\ldots,k\} | \beta_i > 0\}$, $I_\gamma = \{i \in \{1,\ldots,k\} | \gamma_i > 0\}$, $I_\delta = \{i \in \{1,\ldots,k\} | \delta_i > 0\}$, $I_\epsilon = \{i \in \{1,\ldots,k\} | \epsilon_i > 0\}$, $I_B = \{j \in \{1,\ldots,k\} | B_j > 0\}$, $I_C = \{j \in \{1,\ldots,k\} | C_j > 0\}$, $I_D = \{j \in \{1,\ldots,k\} | D_j > 0\}$, and $I_E = \{j \in \{1,\ldots,k\} | E_j > 0\}$. This also proves beneficial later when we adapt

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an unboundedness result from [5] as the author of [5] uses a similar notation.

2. Unboundedness Results Involving Modulo Classes

Here we will present several general theorems which prove unboundedness for systems of two rational difference equations. We feel that it will be helpful for the reader to see some of the special cases which led to theorem 1 even though these cases are eventually subsumed by theorem 1. Here is the first example.

Example 1. *Consider the following system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_2 x_{n-2} + \gamma_2 y_{n-2}}{A + B_2 x_{n-2}}, n = 0, 1, 2, \ldots,$$

$$y_n = \frac{p + \delta_2 x_{n-2} + \epsilon_2 y_{n-2}}{q + E_2 y_{n-2}}, n = 0, 1, 2, \ldots.$$

We assume non-negative parameters and non-negative initial conditions. We further assume the following

(1) $\beta_2, \gamma_2, B_2, \delta_2, \epsilon_2, E_2 > 0$,
(2) $\frac{\gamma_2}{A+B_2} > 2$ *and* $\frac{\delta_2}{q+E_2} > 1$,
(3) $\frac{\alpha+1+\beta_2}{B_2} < 1$ *and* $\frac{p+1+\epsilon_2}{E_2} < 1$,

then the solutions x_n *and* y_n *are unbounded for some non-negative initial conditions.*

Proof. We first prove by induction that under certain non-negative initial conditions $y_{4n} > \max(1, \gamma_2, \delta_2)$ and $x_{4n} < 1$. We choose the initial conditions to provide the base case. Let $y_0 > \max(1, \gamma_2, \delta_2)$ and $x_0 < 1$. Now let us prove the inductive step. Assume $y_{4n-4} > \max(1, \gamma_2, \delta_2)$ and $x_{4n-4} < 1$, then we have

$$x_{4n-2} = \frac{\alpha + \beta_2 x_{4n-4} + \gamma_2 y_{4n-4}}{A + B_2 x_{4n-4}} \geq \frac{\gamma_2 y_{4n-4}}{A + B_2 x_{4n-4}} > \frac{\gamma_2 y_{4n-4}}{A + B_2}.$$

We have assumed that $\frac{\gamma_2}{A+B_2} > 2$ so we have that $x_{4n-2} > 2y_{4n-4}$.
Furthermore we have the following

$$y_{4n-2} = \frac{p + \delta_2 x_{4n-4} + \epsilon_2 y_{4n-4}}{q + E_2 y_{4n-4}} \leq \frac{p + \delta_2 x_{4n-4} + \epsilon_2 y_{4n-4}}{E_2 y_{4n-4}}$$
$$< \frac{p y_{4n-4} + y_{4n-4} + \epsilon_2 y_{4n-4}}{E_2 y_{4n-4}} = \frac{p + 1 + \epsilon_2}{E_2} < 1.$$

Thus $y_{4n-2} < 1$. Now we use these facts to get the following

$$y_{4n} = \frac{p + \delta_2 x_{4n-2} + \epsilon_2 y_{4n-2}}{q + E_2 y_{4n-2}} \geq \frac{\delta_2 x_{4n-2}}{q + E_2 y_{4n-2}} > \frac{\delta_2 x_{4n-2}}{q + E_2}.$$

We have assumed that $\frac{\delta_2}{q+E_2} > 1$ so we have that
$y_{4n} > x_{4n-2} > 2y_{4n-4} > \max(1, \gamma_2, \delta_2)$. Also, since $x_{4n-2} > 2y_{4n-4} > \max(1, \gamma_2, \delta_2)$,we have the following

$$x_{4n} = \frac{\alpha + \beta_2 x_{4n-2} + \gamma_2 y_{4n-2}}{A + B_2 x_{4n-2}} \leq \frac{\alpha x_{4n-2} + \beta_2 x_{4n-2} + x_{4n-2}}{B_2 x_{4n-2}} = \frac{\alpha + \beta_2 + 1}{B_2}.$$

We have assumed that $\frac{\alpha+1+\beta_2}{B_2} < 1$ so we have that $x_{4n} < 1$. Thus we have shown that $y_{4n} > \max(1, \gamma_2, \delta_2)$ and $x_{4n} < 1$ for all $n \in \mathbb{N}$. Notice that we have already shown that this implies that $y_{4n} > x_{4n-2} > 2y_{4n-4}$ for all $n \in \mathbb{N}$ hence $\lim_{n\to\infty} y_{4n} = \infty$ and $\lim_{n\to\infty} x_{4n+2} = \infty$. □

Replacing second order with k^{th} order, the second example proceeds similarly.

Example 2. *Consider the following system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_k x_{n-k} + \gamma_k y_{n-k}}{A + B_k x_{n-k}}, n = 0, 1, 2, \ldots,$$

$$y_n = \frac{p + \delta_k x_{n-k} + \epsilon_k y_{n-k}}{q + E_k y_{n-k}}, n = 0, 1, 2, \ldots.$$

We assume non-negative parameters and non-negative initial conditions. We further assume the following

(1) $\beta_k, \gamma_k, B_k, \delta_k, \epsilon_k, E_k > 0$,
(2) $\frac{\gamma_k}{A+B_k} > 2$ *and* $\frac{\delta_k}{q+E_k} > 1$,
(3) $\frac{\alpha+1+\beta_k}{B_k} < 1$ *and* $\frac{p+1+\epsilon_k}{E_k} < 1$,

then the solutions x_n *and* y_n *are unbounded for some non-negative initial conditions.*

Proof. We first prove by induction that under certain non-negative initial conditions $y_{2kn} > \max(1, \gamma_k, \delta_k)$ and $x_{2kn} < 1$. We choose the initial conditions to provide the base case. Let $y_0 > \max(1, \gamma_k, \delta_k)$ and $x_0 < 1$. Now let us prove the inductive step. Assume $y_{2kn-2k} > \max(1, \gamma_k, \delta_k)$ and $x_{2kn-2k} < 1$, then we have

$$x_{2kn-k} = \frac{\alpha + \beta_k x_{2kn-2k} + \gamma_k y_{2kn-2k}}{A + B_k x_{2kn-2k}} \geq \frac{\gamma_k y_{2kn-2k}}{A + B_k x_{2kn-2k}} > \frac{\gamma_k y_{2kn-2k}}{A + B_k}.$$

We have assumed that $\frac{\gamma_k}{A+B_k} > 2$ so we have that $x_{2kn-k} > 2y_{2kn-2k}$.
Furthermore we have the following

$$y_{2kn-k} = \frac{p + \delta_k x_{2kn-2k} + \epsilon_k y_{2kn-2k}}{q + E_k y_{2kn-2k}} \leq \frac{p + \delta_k x_{2kn-2k} + \epsilon_k y_{2kn-2k}}{E_k y_{2kn-2k}}$$
$$< \frac{p y_{2kn-2k} + y_{2kn-2k} + \epsilon_k y_{2kn-2k}}{E_k y_{2kn-2k}} = \frac{p + 1 + \epsilon_k}{E_k} < 1.$$

Thus $y_{2kn-k} < 1$. Now we use these facts to get the following

$$y_{2kn} = \frac{p + \delta_k x_{2kn-k} + \epsilon_k y_{2kn-k}}{q + E_k y_{2kn-k}} \geq \frac{\delta_k x_{2kn-k}}{q + E_k y_{2kn-k}} > \frac{\delta_k x_{2kn-k}}{q + E_k}.$$

We have assumed that $\frac{\delta_k}{q+E_k} > 1$ so we have that
$y_{2kn} > x_{2kn-k} > 2y_{2kn-2k} > \max(1, \gamma_k, \delta_k)$. Also, since $x_{2kn-k} > 2y_{2kn-2k} > \max(1, \gamma_k, \delta_k)$,we have the following

$$x_{2kn} = \frac{\alpha + \beta_k x_{2kn-k} + \gamma_k y_{2kn-k}}{A + B_k x_{2kn-k}} \leq \frac{\alpha x_{2kn-k} + \beta_k x_{2kn-k} + x_{2kn-k}}{B_k x_{2kn-k}} = \frac{\alpha + \beta_k + 1}{B_k}.$$

We have assumed that $\frac{\alpha+1+\beta_k}{B_k} < 1$ so we have that $x_{2kn} < 1$. Thus we have shown that $y_{2kn} > \max(1, \gamma_k, \delta_k)$ and $x_{2kn} < 1$ for all $n \in \mathbb{N}$. Notice that we have already shown that this implies that $y_{2kn} > x_{2kn-k} > 2y_{2kn-2k}$ for all $n \in \mathbb{N}$ hence $\lim_{n\to\infty} y_{2kn} = \infty$ and $\lim_{n\to\infty} x_{2kn+k} = \infty$. □

Notice that in the example above the key to the proof is that when $n \equiv 0 \mod 2k$ then x_n is small and y_n is large. On the other hand when $n \equiv k \mod 2k$ then x_n is large and y_n is small. So modulo classes play a key role in the above proof though it was unnecessary to mention modulo classes. In the third example the use of modulo classes becomes more explicit.

Example 3. *Consider the following system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_{k-1}x_{n-k+1} + \beta_k x_{n-k} + \gamma_{k-1}y_{n-k+1} + \gamma_k y_{n-k}}{A + B_{k-1}x_{n-k+1} + B_k x_{n-k} + C_k y_{n-k}}, n = 0, 1, 2, \dots,$$

$$y_n = \frac{p + \delta_{k-1}x_{n-k+1} + \delta_k x_{n-k} + \epsilon_{k-1}y_{n-k+1} + \epsilon_k y_{n-k}}{q + D_{k-1}x_{n-k+1} + E_{k-1}y_{n-k+1} + E_k y_{n-k}}, n = 0, 1, 2, \dots,$$

where $k = 3l + 2$ and $l \geq 0$. We assume non-negative parameters and non-negative initial conditions. We further assume the following

(1) $\gamma_{k-1}, B_k, B_{k-1}, \delta_k, E_k, E_{k-1} > 0$,
(2) $\frac{\gamma_{k-1}}{A+B_{k-1}+B_k+C_k} > 2$ *and* $\frac{\delta_k}{q+D_{k-1}+E_k+E_{k-1}} > 1$,
(3) $\frac{\alpha+1+\beta_{k-1}+\beta_k}{\min(B_{k-1},B_k)} + \frac{\gamma_k}{C_k} < 1$ *and* $\frac{p+1+\epsilon_{k-1}+\epsilon_k}{\min(E_{k-1},E_k)} + \frac{\delta_{k-1}}{D_{k-1}} < 1$,
(4) $C_k = 0$ *implies* $\gamma_k = 0$ *and* $D_{k-1} = 0$ *implies* $\delta_{k-1} = 0$,

then the solutions x_n and y_n are unbounded for some non-negative initial conditions.

Proof. We first prove by induction that under certain non-negative initial conditions x_{-m} and y_{-m} where $m \in \{1, \dots, k\}$ so that the following holds. If $-m \equiv -1 \mod 3$, then $y_{-m} > \max(1, \gamma_{k-1}, \delta_k)$ and $x_{-m} < 1$. If $-m \equiv -2 \mod 3$, then $y_{-m} < 1$ and $x_{-m} < 1$. If $-m \equiv -3 \mod 3$, then $y_{-m} < 1$ and $x_{-m} > \max(1, \gamma_{k-1}, \delta_k)$.
Under this choice of initial conditions our solutions $\{x_n\}$ and $\{y_n\}$ have the following properties.

(a) $y_n > \max(1, \gamma_{k-1}, \delta_k)$ and $x_n < 1$ whenever $n \equiv -1 \mod 3$.
(b) $y_n < 1$ and $x_n < 1$ whenever $n \equiv -2 \mod 3$.
(c) $y_n < 1$ and $x_n > \max(1, \gamma_{k-1}, \delta_k)$ whenever $n \equiv -3 \mod 3$.

We prove this using induction on n. Our initial conditions provide the base case. Assume that the statement is true for all $n \leq N - 1$. We show the statement for $n = N$.

This induction proof has three cases. Let us begin by assuming $N \equiv -1 \mod 3$.

Case (a). Since $N \equiv -1 \mod 3$, $N - k = N - 3l - 2 \equiv -3 \mod 3$. So we have that $y_{N-k} < 1$ and $x_{N-k} > \max(1, \gamma_{k-1}, \delta_k)$. Also since $N \equiv -1 \mod 3$, $N - k + 1 = N - 3l - 1 \equiv -2 \mod 3$. So we have that $y_{N-k+1} < 1$ and $x_{N-k+1} < 1$.
From this we demonstrate the desired inequalities $y_N > \max(1, \gamma_{k-1}, \delta_k)$ and $x_N < 1$, given that $N \equiv -1 \mod 3$ holds. Using these facts we get,

$$y_N = \frac{p + \delta_{k-1}x_{N-k+1} + \delta_k x_{N-k} + \epsilon_{k-1}y_{N-k+1} + \epsilon_k y_{N-k}}{q + D_{k-1}x_{N-k+1} + E_{k-1}y_{N-k+1} + E_k y_{N-k}}$$

$$\geq \frac{\delta_k x_{N-k}}{q + D_{k-1}x_{N-k+1} + E_{k-1}y_{N-k+1} + E_k y_{N-k}}$$
$$> \frac{\delta_k x_{N-k}}{q + D_{k-1} + E_{k-1} + E_k} > x_{N-k}$$

since we assumed that $\frac{\delta_k}{q+D_{k-1}+E_k+E_{k-1}} > 1$. Now since $x_{N-k} > \max(1, \gamma_{k-1}, \delta_k)$, $y_N > \max(1, \gamma_{k-1}, \delta_k)$.

Now we show that $x_N < 1$. We have

$$x_N = \frac{\alpha + \beta_{k-1}x_{N-k+1} + \beta_k x_{N-k} + \gamma_{k-1}y_{N-k+1} + \gamma_k y_{N-k}}{A + B_{k-1}x_{N-k+1} + B_k x_{N-k} + C_k y_{N-k}}$$
$$< \frac{\alpha}{B_k x_{N-k}} + \frac{\gamma_{k-1}y_{N-k+1}}{B_k x_{N-k}} + \frac{\beta_{k-1} + \beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k}$$
$$< \frac{\alpha+1}{B_k} + \frac{\beta_{k-1} + \beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k} \leq \frac{\alpha + 1 + \beta_{k-1} + \beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k} < 1$$

since we assumed that $\frac{\alpha+1+\beta_{k-1}+\beta_k}{\min(B_k,B_{k-1})} + \frac{\gamma_k}{C_k} < 1$. This finishes case (a).

Case (b). We now assume that $N \equiv -2 \mod 3$.
Since $N \equiv -2 \mod 3$, $N - k = N - 3l - 2 \equiv -4 \equiv -1 \mod 3$. So we have that $y_{N-k} > \max(1, \gamma_{k-1}, \delta_k)$ and $x_{N-k} < 1$. Also since $N \equiv -2 \mod 3$, $N - k + 1 = N - 3l - 1 \equiv -3 \mod 3$. So we have that $y_{N-k+1} < 1$ and $x_{N-k+1} > \max(1, \gamma_{k-1}, \delta_k)$.
From this we demonstrate the desired inequalities $y_N < 1$ and $x_N < 1$, given that $N \equiv -2 \mod 3$ holds. Hence

$$y_N = \frac{p + \delta_{k-1}x_{N-k+1} + \delta_k x_{N-k} + \epsilon_{k-1}y_{N-k+1} + \epsilon_k y_{N-k}}{q + D_{k-1}x_{N-k+1} + E_{k-1}y_{N-k+1} + E_k y_{N-k}}$$
$$< \frac{p}{E_k y_{N-k}} + \frac{\delta_k x_{N-k}}{E_k y_{N-k}} + \frac{\epsilon_{k-1} + \epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}}$$
$$< \frac{p+1}{E_k} + \frac{\epsilon_{k-1} + \epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}} \leq \frac{p + 1 + \epsilon_{k-1} + \epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}} < 1$$

since we assumed that $\frac{p+1+\epsilon_{k-1}+\epsilon_k}{\min(E_{k-1},E_k)} + \frac{\delta_{k-1}}{D_{k-1}} < 1$.

Now we show that $x_N < 1$. We have

$$x_N = \frac{\alpha + \beta_{k-1}x_{N-k+1} + \beta_k x_{N-k} + \gamma_{k-1}y_{N-k+1} + \gamma_k y_{N-k}}{A + B_{k-1}x_{N-k+1} + B_k x_{N-k} + C_k y_{N-k}}$$
$$< \frac{\alpha}{B_{k-1}x_{N-k+1}} + \frac{\gamma_{k-1}y_{N-k+1}}{B_{k-1}x_{N-k+1}} + \frac{\beta_{k-1} + \beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k}$$
$$< \frac{\alpha+1}{B_{k-1}} + \frac{\beta_{k-1} + \beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k} \leq \frac{\alpha + 1 + \beta_{k-1} + \beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k} < 1$$

since we assumed that $\frac{\alpha+1+\beta_{k-1}+\beta_k}{\min(B_k,B_{k-1})} + \frac{\gamma_k}{C_k} < 1$. This finishes case (b).

Case (c). We now assume that $N \equiv -3 \mod 3$.
Since $N \equiv -3 \mod 3$, $N - k = N - 3l - 2 \equiv -5 \equiv -2 \mod 3$. So we have that $y_{N-k} < 1$ and $x_{N-k} < 1$. Also since $N \equiv -3 \mod 3$, $N-k+1 = N-3l-1 \equiv -4 \equiv -1 \mod 3$. So we have that $y_{N-k+1} > \max(1, \gamma_{k-1}, \delta_k)$ and $x_{N-k+1} < 1$.
From this we demonstrate the desired inequalities $y_N < 1$ and $x_N > \max(1, \gamma_{k-1}, \delta_k)$, given that $N \equiv -3 \mod 3$ holds. Hence

$$\begin{aligned} y_N &= \frac{p + \delta_{k-1}x_{N-k+1} + \delta_k x_{N-k} + \epsilon_{k-1}y_{N-k+1} + \epsilon_k y_{N-k}}{q + D_{k-1}x_{N-k+1} + E_{k-1}y_{N-k+1} + E_k y_{N-k}} \\ &< \frac{p}{E_{k-1}y_{N-k+1}} + \frac{\delta_k x_{N-k}}{E_{k-1}y_{N-k+1}} + \frac{\epsilon_{k-1} + \epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}} \\ &< \frac{p+1}{E_{k-1}} + \frac{\epsilon_{k-1} + \epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}} \leq \frac{p + 1 + \epsilon_{k-1} + \epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}} < 1 \end{aligned}$$

since we assumed that $\frac{p+1+\epsilon_{k-1}+\epsilon_k}{\min(E_{k-1},E_k)} + \frac{\delta_{k-1}}{D_{k-1}} < 1$.

Now we show that $x_N > max(1, \gamma_{k-1}, \delta_k)$. We have

$$\begin{aligned} x_N &= \frac{\alpha + \beta_{k-1}x_{N-k+1} + \beta_k x_{N-k} + \gamma_{k-1}y_{N-k+1} + \gamma_k y_{N-k}}{A + B_{k-1}x_{N-k+1} + B_k x_{N-k} + C_k y_{N-k}} \\ &\geq \frac{\gamma_{k-1}y_{N-k+1}}{A + B_{k-1}x_{N-k+1} + B_k x_{N-k} + C_k y_{N-k}} \\ &> \frac{\gamma_{k-1}y_{N-k+1}}{A + B_{k-1} + B_k + C_k} > 2y_{N-k+1} \end{aligned}$$

since we assumed that $\frac{\gamma_{k-1}}{A+B_{k-1}+B_k+C_k} > 2$. Now since $y_{N-k+1} > \max(1, \gamma_{k-1}, \delta_k)$, $x_N > \max(1, \gamma_{k-1}, \delta_k)$.

We now conclude through proof by induction that $\lim_{n\to\infty} y_{(2k-1)n+2} = \infty$ and $\lim_{n\to\infty} x_{(2k-1)n+k+1} = \infty$.

We first see that

$$\begin{aligned} y_{(2k-1)n+2} &= \frac{p + \delta_{k-1}x_{(2k-1)n+3-k} + \delta_k x_{(2k-1)n+2-k} + \epsilon_{k-1}y_{(2k-1)n+3-k} + \epsilon_k y_{(2k-1)n+2-k}}{q + D_{k-1}x_{(2k-1)n+3-k} + E_{k-1}y_{(2k-1)n+3-k} + E_k y_{(2k-1)n+2-k}} \\ &\geq \frac{\delta_k x_{(2k-1)n+2-k}}{q + D_{k-1}x_{(2k-1)n+3-k} + E_{k-1}y_{(2k-1)n+3-k} + E_k y_{(2k-1)n+2-k}} \\ &> \frac{\delta_k x_{(2k-1)n+2-k}}{q + D_{k-1} + E_{k-1} + E_k} \end{aligned}$$

since $(2k-1)n + 2 - k \equiv -3 \mod 3$ and since $(2k-1)n + 3 - k \equiv -2 \mod 3$. Also note that $y_{(2k-1)n+2} > x_{(2k-1)n+2-k}$, since $\frac{\delta_k}{q+D_{k-1}+E_{k-1}+E_k} > 1$ by assumption (2).

Now,

$$y_{(2k-1)n+2} > x_{(2k-1)n+2-k} \geq \frac{\gamma_{k-1}y_{(2k-1)(n-1)+2}}{A + B_{k-1}x_{(2k-1)(n-1)+2} + B_k x_{(2k-1)(n-1)+1} + C_k y_{(2k-1)(n-1)+1}}$$

$$> \left(\frac{\gamma_{k-1}}{A + B_{k-1} + B_k + C_k}\right)(y_{(2k-1)(n-1)+2})$$

since $(2k-1)(n-1)+2 \equiv -1 \mod 3$ and since $(2k-1)(n-1)+1 \equiv -2 \mod 3$. From assumption (2), we have that $\frac{\gamma_{k-1}}{A+B_{k-1}+B_k+C_k} > 2$. So that $y_{(2k-1)n+2} > x_{(2k-1)n+2-k} > 2y_{(2k-1)(n-1)+2}$ for all $n \in \mathbb{N}$, which proves that $\lim_{n\to\infty} y_{(2k-1)n+2} = \infty$. Also, since $y_{(2k-1)n+2} > x_{(2k-1)n+2-k} > 2y_{(2k-1)(n-1)+2}$, for all $n \in \mathbb{N}$, $x_{(2k-1)n+k+1} > 2y_{(2k-1)n+2} > 2x_{(2k-1)n+2-k}$, for all $n \in \mathbb{N}$, which proves that $\lim_{n\to\infty} x_{(2k-1)n+k+1} = \infty$.

□

Since we shall prove unboundedness via use of modulo classes let us first introduce some new notation. Given a set $S \subset \mathbb{Z}$ we let S^a denote the set comprised of the residues modulo a of the elements of our set S. Written another way $S^a = \{x \in \{0, \dots, a-1\} | x \equiv s \mod a \text{ for some } s \in S\}$. We use this notation to keep track of how the sets of residues modulo a of our indices of our system of difference equations behave.

Theorem 1. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}, n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}, n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Suppose that there exists a and b such that all of the following hold,

(1) $0 \le b < a$,
(2) $I_B^a = \{1, \dots, a-1\}$,
(3) $I_E^a = \{1, \dots, a-1\}$,
(4) $(I_\gamma \setminus I_C)^a = \{b\}$,
(5) $(I_\delta \setminus I_D)^a = \{-b \bmod a\}$,
(6) $(I_\beta \setminus I_B)^a \subset \{0\}$,
(7) $(I_\epsilon \setminus I_E)^a \subset \{0\}$,
(8) $b \notin I_C^a$,
(9) $-b \bmod a \notin I_D^a$.

Also assume the following

(1) $\frac{\sum_{i\in I_\delta \setminus I_D} \delta_i}{q+\sum_{j=1}^k D_j + \sum_{j=1}^k E_j} > 1$,
(2) $\frac{\sum_{i\in I_\gamma \setminus I_C} \gamma_i}{A+\sum_{j=1}^k B_j + \sum_{j=1}^k C_j} > 2$,
(3) $\sum_{i\in I_C} \frac{\gamma_i}{C_i} + \frac{\alpha+1+\sum_{i=1}^k \beta_i}{\min_{j\in I_B}(B_j)} < 1$,
(4) $\sum_{i\in I_D} \frac{\delta_i}{D_i} + \frac{p+1+\sum_{i=1}^k \epsilon_i}{\min_{j\in I_E}(E_j)} < 1$.

then for some choice of initial conditions $\lim_{n\to\infty} x_{an+b} = \infty$ *and* $\lim_{n\to\infty} y_{an} = \infty$.

Proof. We let our initial conditions provide the base case and use strong induction on N to prove that $x_{aN+b} > \max(1, \sum_{i=1}^k \delta_i, \sum_{i=1}^k \gamma_i)$,

$y_{aN} > \max(1, \sum_{i=1}^{k} \delta_i, \sum_{i=1}^{k} \gamma_i)$,
$x_{aN+s}, y_{aN+r} < 1$, for $s, r \in \{0, \ldots, a-1\}$ with $s \neq b$ and $r \neq 0$.
So assume that the following holds for $n < N$,
$x_{an+b} > \max(1, \sum_{i=1}^{k} \delta_i, \sum_{i=1}^{k} \gamma_i)$,
$y_{an} > \max(1, \sum_{i=1}^{k} \delta_i, \sum_{i=1}^{k} \gamma_i)$,
$x_{an+s}, y_{an+r} < 1$, for $s, r \in \{0, \ldots, a-1\}$ with $s \neq b$ and $r \neq 0$.
Then we have

$$\begin{aligned} x_{aN+b} &= \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{aN+b-i} + \sum_{i=1}^{k} \gamma_i y_{aN+b-i}}{A + \sum_{j=1}^{k} B_j x_{aN+b-j} + \sum_{j=1}^{k} C_j y_{aN+b-j}} \\ &\geq \frac{\sum_{i \in I_\gamma \setminus I_C} \gamma_i y_{aN+b-i}}{A + \sum_{j=1}^{k} B_j x_{aN+b-j} + \sum_{j=1}^{k} C_j y_{aN+b-j}} \\ &\geq \frac{(\sum_{i \in I_\gamma \setminus I_C} \gamma_i) \min_{i \in I_\gamma \setminus I_C}(y_{aN+b-i})}{A + \sum_{j=1}^{k} B_j x_{aN+b-j} + \sum_{j=1}^{k} C_j y_{aN+b-j}}. \end{aligned}$$

Now since $b \notin I_C^a$ and $I_B^a = \{1, \ldots, a-1\}$ we have that $aN + b - j_1 \not\equiv 0 \mod a$ for all $j_1 \in I_C$ and $aN + b - j_2 \not\equiv b \mod a$ for all $j_2 \in I_B$, thus we get

$$x_{aN+b} > \frac{(\sum_{i \in I_\gamma \setminus I_C} \gamma_i) \min_{i \in I_\gamma \setminus I_C}(y_{aN+b-i})}{A + \sum_{j=1}^{k} B_j + \sum_{j=1}^{k} C_j}.$$

Now since we have assumed that $\frac{\sum_{i \in I_\gamma \setminus I_C} \gamma_i}{A + \sum_{j=1}^{k} B_j + \sum_{j=1}^{k} C_j} > 2$, we get

$$x_{aN+b} > 2 \min_{i \in I_\gamma \setminus I_C} (y_{aN+b-i}). \tag{1}$$

Since $(I_\gamma \setminus I_C)^a = \{b\}$, $aN + b - i \equiv 0 \mod a$ for all $i \in I_\gamma \setminus I_C$, thus

$$x_{aN+b} > 2 \min_{i \in I_\gamma \setminus I_C} (y_{aN+b-i}) > 2 \max(1, \sum_{i=1}^{k} \delta_i, \sum_{i=1}^{k} \gamma_i).$$

Also we have the following

$$\begin{aligned} y_{aN} &= \frac{p + \sum_{i=1}^{k} \delta_i x_{aN-i} + \sum_{i=1}^{k} \epsilon_i y_{aN-i}}{q + \sum_{j=1}^{k} D_j x_{aN-j} + \sum_{j=1}^{k} E_j y_{aN-j}} \\ &\geq \frac{\sum_{i \in I_\delta \setminus I_D} \delta_i x_{aN-i}}{q + \sum_{j=1}^{k} D_j x_{aN-j} + \sum_{j=1}^{k} E_j y_{aN-j}} \\ &\geq \frac{(\sum_{i \in I_\delta \setminus I_D} \delta_i) \min_{i \in I_\delta \setminus I_D}(x_{aN-i})}{q + \sum_{j=1}^{k} D_j x_{aN-j} + \sum_{j=1}^{k} E_j y_{aN-j}}. \end{aligned}$$

Now since $-b \mod a \notin I_D^a$ and $I_E^a = \{1, \ldots, a-1\}$ we have that $aN - j_1 \not\equiv 0 \mod a$ for all $j_1 \in I_E$ and $aN - j_2 \not\equiv b \mod a$ for all $j_2 \in I_D$, thus we get

$$y_{aN} > \frac{(\sum_{i \in I_\delta \setminus I_D} \delta_i) \min_{i \in I_\delta \setminus I_D}(x_{aN-i})}{q + \sum_{j=1}^{k} D_j + \sum_{j=1}^{k} E_j}.$$

Now since we have assumed that $\frac{\sum_{i\in I_\delta\setminus I_D}\delta_i}{q+\sum_{j=1}^k D_j+\sum_{j=1}^k E_j}>1$, we get

$$y_{aN}>\min_{i\in I_\delta\setminus I_D}(x_{aN-i}). \tag{2}$$

Since $(I_\delta\setminus I_D)^a=\{-b \mod a\}$, $aN-i\equiv b \mod a$ for all $i\in I_\delta\setminus I_D$, thus

$$y_{aN}>\min_{i\in I_\delta\setminus I_D}(x_{aN-i})>\max(1,\sum_{i=1}^k\delta_i,\sum_{i=1}^k\gamma_i).$$

We now prove the remaining inequalities. For $s\in\{0,\dots,a-1\}$ with $s\neq b$,

$$x_{aN+s}=\frac{\alpha+\sum_{i=1}^k\beta_i x_{aN+s-i}+\sum_{i=1}^k\gamma_i y_{aN+s-i}}{A+\sum_{j=1}^k B_j x_{aN+s-j}+\sum_{j=1}^k C_j y_{aN+s-j}}$$

$$\le\frac{\alpha+\sum_{i=1}^k\beta_i x_{aN+s-i}+\sum_{i\in I_\gamma\setminus I_C}\gamma_i y_{aN+s-i}}{A+\sum_{j=1}^k B_j x_{aN+s-j}+\sum_{j=1}^k C_j y_{aN+s-j}}+\sum_{i\in I_C}\frac{\gamma_i y_{aN+s-i}}{A+\sum_{j=1}^k B_j x_{aN+s-j}+\sum_{j=1}^k C_j y_{aN+s-j}}$$

$$\le\frac{\alpha+\sum_{i=1}^k\beta_i x_{aN+s-i}+\sum_{i\in I_\gamma\setminus I_C}\gamma_i y_{aN+s-i}}{\min_{j\in I_B}(B_j)\max_{j\in I_B}(x_{aN+s-j})}+\sum_{i\in I_C}\frac{\gamma_i}{C_i}.$$

Since $(I_\beta\setminus I_B)^a\subset\{0\}$ we have that for all $i\in I_\beta$, $i\in I_B$ or $i\in\{z\in\mathbb{Z}|z\equiv 0 \mod a\}$. Thus for all $i\in I_\beta$ either $x_{aN+s-i}\le\max_{j\in I_B}(x_{aN+s-j})$, or $x_{aN+s-i}<1$. Furthermore since $I_B^a=\{1,\dots,a-1\}$ there exists $j\in I_B$ so that $aN+s-j\equiv b \mod a$. Thus $\max_{j\in I_B}(x_{aN+s-j})>1$ and $\max_{j\in I_B}(x_{aN+s-j})>\sum_{i=1}^k\gamma_i$. To be clear notice that this means $\max_{i\in I_\beta}(x_{aN+s-i})\le\max_{j\in I_B}(x_{aN+s-j})$. So we get

$$x_{aN+s}\le\frac{\sum_{i=1}^k\beta_i}{\min_{j\in I_B}(B_j)}+\frac{\alpha+\sum_{i\in I_\gamma\setminus I_C}\gamma_i y_{aN+s-i}}{\min_{j\in I_B}(B_j)\max_{j\in I_B}(x_{aN+s-j})}+\sum_{i\in I_C}\frac{\gamma_i}{C_i}$$

$$\le\frac{\alpha+\sum_{i=1}^k\beta_i}{\min_{j\in I_B}(B_j)}+\frac{\sum_{i\in I_\gamma\setminus I_C}\gamma_i y_{aN+s-i}}{\min_{j\in I_B}(B_j)\max_{j\in I_B}(x_{aN+s-j})}+\sum_{i\in I_C}\frac{\gamma_i}{C_i}.$$

Now since $(I_\gamma\setminus I_C)^a=\{b\}$, $aN+s-i\not\equiv 0 \mod a$ for all $i\in I_\gamma\setminus I_C$. Thus $y_{aN+s-i}<1$ for all $i\in I_\gamma\setminus I_C$. Hence

$$x_{aN+s}<\frac{\alpha+\sum_{i=1}^k\beta_i}{\min_{j\in I_B}(B_j)}+\frac{\sum_{i\in I_\gamma\setminus I_C}\gamma_i}{\min_{j\in I_B}(B_j)\max_{j\in I_B}(x_{aN+s-j})}+\sum_{i\in I_C}\frac{\gamma_i}{C_i}$$

$$<\frac{\alpha+1+\sum_{i=1}^k\beta_i}{\min_{j\in I_B}(B_j)}+\sum_{i\in I_C}\frac{\gamma_i}{C_i}<1.$$

Now for $r\in\{0,\dots,a-1\}$ with $r\neq 0$,

$$y_{aN+r}=\frac{p+\sum_{i=1}^k\delta_i x_{aN+r-i}+\sum_{i=1}^k\epsilon_i y_{aN+r-i}}{q+\sum_{j=1}^k D_j x_{aN+r-j}+\sum_{j=1}^k E_j y_{aN+r-j}}$$

$$\leq \frac{p+\sum_{i\in I_\delta\setminus I_D}\delta_i x_{aN+r-i}+\sum_{i=1}^k \epsilon_i y_{aN+r-i}}{q+\sum_{j=1}^k D_j x_{aN+r-j}+\sum_{j=1}^k E_j y_{aN+r-j}}+\sum_{i\in I_D}\frac{\delta_i x_{aN+r-i}}{q+\sum_{j=1}^k D_j x_{aN+r-j}+\sum_{j=1}^k E_j y_{aN+r-j}}$$

$$\leq \frac{p+\sum_{i\in I_\delta\setminus I_D}\delta_i x_{aN+r-i}+\sum_{i=1}^k \epsilon_i y_{aN+r-i}}{\min_{j\in I_E}(E_j)\max_{j\in I_E}(y_{aN+r-j})}+\sum_{i\in I_D}\frac{\delta_i}{D_i}.$$

Since $(I_\epsilon \setminus I_E)^a \subset \{0\}$ we have that for all $i \in I_\epsilon$, $i \in I_E$ or $i \in \{z \in \mathbb{Z} | z \equiv 0 \mod a\}$. Thus for all $i \in I_\epsilon$ either $y_{aN+r-i} \leq \max_{j\in I_E}(y_{aN+r-j})$, or $y_{aN+r-i} < 1$. Furthermore since $I_E^a = \{1, \dots, a-1\}$ there exists $j \in I_E$ so that $aN + r - j \equiv 0 \mod a$. Thus $\max_{j\in I_E}(y_{aN+r-j}) > 1$ and $\max_{j\in I_E}(y_{aN+r-j}) > \sum_{i=1}^k \delta_i$. To be clear this means $\max_{i\in I_\epsilon}(y_{aN+r-i}) \leq \max_{j\in I_E}(y_{aN+r-j})$. So we get

$$y_{aN+r} \leq \frac{\sum_{i=1}^k \epsilon_i}{\min_{j\in I_E}(E_j)}+\frac{p+\sum_{i\in I_\delta\setminus I_D}\delta_i x_{aN+r-i}}{\min_{j\in I_E}(E_j)\max_{j\in I_E}(y_{aN+r-j})}+\sum_{i\in I_D}\frac{\delta_i}{D_i}$$

$$\leq \frac{p+\sum_{i=1}^k \epsilon_i}{\min_{j\in I_E}(E_j)}+\frac{\sum_{i\in I_\delta\setminus I_D}\delta_i x_{aN+r-i}}{\min_{j\in I_E}(E_j)\max_{j\in I_E}(y_{aN+r-j})}+\sum_{i\in I_D}\frac{\delta_i}{D_i}.$$

Now since $(I_\delta \setminus I_D)^a = \{-b \mod a\}$, $aN + r - i \not\equiv b \mod a$ for all $i \in I_\delta \setminus I_D$. Thus $x_{aN+r-i} < 1$ for all $i \in I_\delta \setminus I_D$. Hence

$$x_{aN+s} < \frac{p+\sum_{i=1}^k \epsilon_i}{\min_{j\in I_E}(E_j)}+\frac{\sum_{i\in I_\delta\setminus I_D}\delta_i}{\min_{j\in I_E}(E_j)\max_{j\in I_E}(y_{aN+r-j})}+\sum_{i\in I_D}\frac{\delta_i}{D_i}$$

$$< \frac{p+1+\sum_{i=1}^k \epsilon_i}{\min_{j\in I_E}(E_j)}+\sum_{i\in I_D}\frac{\delta_i}{D_i} < 1.$$

Thus we have completed the induction proof and hence
$x_{aN+b} > \max(1, \sum_{i=1}^k \delta_i, \sum_{i=1}^k \gamma_i)$,
$y_{aN} > \max(1, \sum_{i=1}^k \delta_i, \sum_{i=1}^k \gamma_i)$,
$x_{aN+s}, y_{aN+r} < 1$, for $s, r \in \{0, \dots, a-1\}$ with $s \neq b$ and $r \neq 0$.
for all $N \in \mathbb{N}$. Now recall from inequalities (1) and (2) that we have now shown

$$x_{aN+b} > 2\min_{i\in I_\gamma\setminus I_C}(y_{aN+b-i}),$$

and

$$y_{aN} > \min_{i\in I_\delta\setminus I_D}(x_{aN-i}),$$

for all $N \in \mathbb{N}$. Here we use substitution and arrive at the following inequalities

$$x_{aN+b} > 2\min_{u\in U}(x_{aN+b-u}),$$

and

$$y_{aN} > 2\min_{u\in U}(y_{aN-u}),$$

for all $N \in \mathbb{N}$, where $U = \{i_1 + i_2 | i_1 \in I_\gamma \setminus I_C \text{ and } i_2 \in I_\delta \setminus I_D\}$. Using the fact that $(I_\gamma \setminus I_C)^a = \{b\}$ and $(I_\delta \setminus I_D)^a = \{-b \mod a\}$ we find that $U \subset \{a\eta | \eta \in \{1, \ldots, 2\lceil \frac{k}{a} \rceil\}\}$ Thus the following inequalities hold for all $n \geq 2k$.

$$x_{an+b} > 2 \min_{i \in \{1,\ldots,2\lceil \frac{k}{a} \rceil\}} (x_{a(n-i)+b}),$$

and

$$y_{an} > 2 \min_{i \in \{1,\ldots,2\lceil \frac{k}{a} \rceil\}} (y_{a(n-i)}).$$

Now let us make the following change of variables $x_{an+b} = w_n$ and $y_{an} = v_n$, thus we get the following difference inequalities

$$w_n > 2 \min_{i \in \{1,\ldots,2\lceil \frac{k}{a} \rceil\}} (w_{n-i}),$$

and

$$v_n > 2 \min_{i \in \{1,\ldots,2\lceil \frac{k}{a} \rceil\}} (v_{n-i}).$$

for all $n \geq 2k$. Thus using theorem 3 in [4] we get

$$\min(w_{n-1}, \ldots, w_{n-2\lceil \frac{k}{a} \rceil}) \geq 2^{\lfloor \frac{n-2k}{2\lceil \frac{k}{a} \rceil} \rfloor} \min_{i \in \{1,\ldots,2\lceil \frac{k}{a} \rceil\}} (w_{2k-i}),$$

and

$$\min(v_{n-1}, \ldots, v_{n-2\lceil \frac{k}{a} \rceil}) \geq 2^{\lfloor \frac{n-2k}{2\lceil \frac{k}{a} \rceil} \rfloor} \min_{i \in \{1,\ldots,2\lceil \frac{k}{a} \rceil\}} (v_{2k-i}).$$

Hence we have $\lim_{n\to\infty} w_n = \infty$ and $\lim_{n\to\infty} v_n = \infty$. Thus $\lim_{n\to\infty} x_{an+b} = \infty$ and $\lim_{n\to\infty} y_{an} = \infty$. □

We have just presented a general unboundedness result for systems of rational difference equations. Notice that examples 1 and 2 are subsumed by theorem 1 after a change of variables. We prove above that, when the hypotheses are satisfied, both $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are unbounded. However there are known special cases where $\{x_n\}_{n=0}^{\infty}$ is unbounded and $\{y_n\}_{n=0}^{\infty}$ is bounded above by a positive constant, and vice versa. In fact it is possible to sometimes apply similar techniques to those presented above in these cases. This is what motivates the following theorem. We prove the result for the case where $\{x_n\}_{n=0}^{\infty}$ is unbounded and $\{y_n\}_{n=0}^{\infty}$ is bounded above by a positive constant. In the other case we advise the reader to make a change of variables.

Theorem 2. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Suppose that there exist initial conditions $y_0, \ldots, y_{-k+1}$ and that there exists $M > 0$ so that $y_n \leq M$ for *all $n > -k$ and for all choices of initial conditions $x_0, \dots, x_{-k+1}$. Further suppose that there exists a such that all of the following hold,*

(1) $I_B^a = \{1, \dots, a-1\}$,
(2) $(I_\beta \setminus I_B)^a = \{0\}$,

Also assume the following

(1) $\frac{\sum_{i \in I_\beta \setminus I_B} \beta_i}{A + \sum_{j=1}^k B_j + \sum_{j=1}^k C_j M} > 2$,
(2) $\frac{\alpha + 1 + \sum_{i \in I_B} \beta_i}{\min_{j \in I_B}(B_j)} < 1$.

then for some choice of initial conditions $\lim_{n \to \infty} x_{an} = \infty$.

Proof. We let our initial conditions provide the base case and use strong induction on N to prove that $x_{aN} > \max(1, \sum_{i=1}^k \beta_i + \sum_{i=1}^k \gamma_i M)$,
$x_{aN+s} < 1$, for $s \in \{1, \dots, a-1\}$.
So assume that the following holds for $n < N$,
$x_{an} > \max(1, \sum_{i=1}^k \beta_i + \sum_{i=1}^k \gamma_i M)$,
$x_{an+s} < 1$, for $s \in \{1, \dots, a-1\}$.
Then we have

$$x_{aN} = \frac{\alpha + \sum_{i=1}^k \beta_i x_{aN-i} + \sum_{i=1}^k \gamma_i y_{aN-i}}{A + \sum_{j=1}^k B_j x_{aN-j} + \sum_{j=1}^k C_j y_{aN-j}}$$
$$\geq \frac{\sum_{i \in I_\beta \setminus I_B} \beta_i x_{aN-i}}{A + \sum_{j=1}^k B_j x_{aN-j} + \sum_{j=1}^k C_j y_{aN-j}}.$$

Since $I_B^a = \{1, \dots, a-1\}$ and $y_n \leq M$ we have,

$$x_{aN} \geq \frac{\sum_{i \in I_\beta \setminus I_B} \beta_i x_{aN-i}}{A + \sum_{j=1}^k B_j + \sum_{j=1}^k C_j M}$$
$$\geq \frac{(\sum_{i \in I_\beta \setminus I_B} \beta_i)(\min_{i \in I_\beta \setminus I_B}(x_{aN-i}))}{A + \sum_{j=1}^k B_j + \sum_{j=1}^k C_j M}.$$

Now since we have assumed that $\frac{\sum_{i \in I_\beta \setminus I_B} \beta_i}{A + \sum_{j=1}^k B_j + \sum_{j=1}^k C_j M} > 2$, we get

$$x_{aN} > 2 \min_{i \in I_\beta \setminus I_B} (x_{aN-i}). \tag{3}$$

Since $(I_\beta \setminus I_B)^a = \{0\}$, $aN - i \equiv 0 \mod a$ for all $i \in I_\beta \setminus I_B$, thus

$$x_{aN} > 2 \min_{i \in I_\beta \setminus I_B} (x_{aN-i}) > 2 \max(1, \sum_{i=1}^k \beta_i + \sum_{i=1}^k \gamma_i M).$$

We now prove the remaining inequality. For $s \in \{1, \dots, a-1\}$,

$$x_{aN+s} = \frac{\alpha + \sum_{i=1}^k \beta_i x_{aN+s-i} + \sum_{i=1}^k \gamma_i y_{aN+s-i}}{A + \sum_{j=1}^k B_j x_{aN+s-j} + \sum_{j=1}^k C_j y_{aN+s-j}}$$

$$\leq \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{aN+s-i} + \sum_{i=1}^{k} \gamma_i M}{\sum_{j=1}^{k} B_j x_{aN+s-j}}.$$

Since $(I_\beta \setminus I_B)^a = \{0\}$ we have that for all $i \in I_\beta \setminus I_B$, $x_{aN+s-i} < 1$. So we have

$$x_{aN+s} \leq \frac{\sum_{i \in I_B} \beta_i}{\min_{j \in I_B}(B_j)} + \frac{\alpha + \sum_{i \in I_\beta \setminus I_B} \beta_i + \sum_{i=1}^{k} \gamma_i M}{\min_{j \in I_B}(B_j) \max_{j \in I_B}(x_{aN+s-j})}.$$

Since $I_B^a = \{1, \ldots, a-1\}$ there exists $j \in I_B$ so that $aN + s - j \equiv 0 \mod a$. Thus $\max_{j \in I_B}(x_{aN+s-j}) > 1$ and $\max_{j \in I_B}(x_{aN+s-j}) > \sum_{i=1}^{k} \beta_i + \sum_{i=1}^{k} \gamma_i M$. So we get

$$x_{aN+s} < \frac{\alpha + 1 + \sum_{i \in I_B} \beta_i}{\min_{j \in I_B}(B_j)}.$$

Thus we have completed the induction proof and hence
$x_{aN} > \max(1, \sum_{i=1}^{k} \beta_i + \sum_{i=1}^{k} \gamma_i M)$,
$x_{aN+s} < 1$, for $s \in \{1, \ldots, a-1\}$.
for all $N \in \mathbb{N}$. Now recall from the inequality (3) that we have now shown

$$x_{aN} > 2 \min_{i \in I_\beta \setminus I_B} (x_{aN-i}),$$

for all $N \in \mathbb{N}$. So we make a change of variables $x_{an} = w_n$ and we get the following difference inequality

$$w_n > 2 \min_{i \in \{1, \ldots, \lfloor \frac{k}{a} \rfloor\}} (w_{n-i}).$$

for all $n \geq k$. Thus using theorem 3 in [4] we get

$$\min(w_{n-1}, \ldots, w_{n-\lfloor \frac{k}{a} \rfloor}) \geq 2^{\lfloor \frac{n-k}{\lfloor \frac{k}{a} \rfloor} \rfloor} \min_{i \in \{1, \ldots, \lfloor \frac{k}{a} \rfloor\}} (w_{k-i}).$$

Hence we have $\lim_{n \to \infty} w_n = \infty$, thus $\lim_{n \to \infty} x_{an} = \infty$. □

3. Adapting an unboundedness result to systems

Let us draw our attention to theorem 2 case (iii) of [5]. To prove this result the author separates the integers into two sets $A = \{n \in \mathbb{Z} : \gcd(I_\beta) | n\}$ and $B = \mathbb{Z} \setminus A$. The author then proves via induction that for proper choice of initial conditions, whenever $n \in A$ then $x_n > 0$, and whenever $n \in B$ then $x_n = 0$. The key here is that parameters are chosen in a way that makes such a proof possible. We wish to adapt such a result so that it might apply to systems. Thus it is important to choose our parameters so that a similar idea holds. The first thing which comes to mind is to require that there does not exist $j \in I_B \cup I_C \cup I_D \cup I_E$ so that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon) | j$. This motivates the following theorem.

Theorem 3. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, n \in \mathbb{N},$$

$$y_n = \frac{\sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}, n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions.
Further assume that $q, A > 0$ *and that one of the following holds,*

(1) $A < \sum_{i=1}^k \beta_i$, *and* $I_\delta \neq \emptyset$
(2) $q < \sum_{i=1}^k \epsilon_i$, *and* $I_\gamma \neq \emptyset$

Also suppose that there does not exist $j \in I_B \cup I_C \cup I_D \cup I_E$ *so that* $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|j$. *Then unbounded solutions exist for both* $\{x_n\}_{n=0}^\infty$ *and* $\{y_n\}_{n=0}^\infty$ *for some choice of initial conditions.*

Proof. Choose initial conditions x_{-m} and y_{-m} where $m \in \{0, \ldots, k-1\}$ so that $x_{-m} = 1 = y_{-m}$ if $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|m$ and $x_{-m} = 0 = y_{-m}$ otherwise.
Under this choice of initial conditions, $\{x_n\}$ and $\{y_n\}$ have the property that $x_n > 0$ and $y_n > 0$ whenever $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|n$ and $x_n = 0 = y_n$ otherwise. We prove this using induction on n, our initial conditions provide the base case. Assume that the statement is true for all $n \leq N-1$. We show the statement for $n = N$.
This argument has four cases. First assume $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$ then both denominators are clearly non-zero since we assumed $A, q > 0$. Since we assumed $\sum_{i=1}^k \beta_i + \sum_{i=1}^k \gamma_i > 0$ we know that either there exists $i \in I_\beta$ so that $\beta_i > 0$, or there exists $i \in I_\gamma$ so that $\gamma_i > 0$. It is sufficient to show $x_{N-i} > 0$ and $y_{N-i} > 0$. However since $i \in I_\beta \cup I_\gamma$ it follows that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|i$. Thus $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N-i$. Hence by our induction hypothesis $x_{N-i} > 0$ and $y_{N-i} > 0$. Thus $x_N > 0$. We have shown the first case.
Again assume $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$ then both denominators are clearly non-zero since we assumed $A, q > 0$. Since we assumed $\sum_{i=1}^k \delta_i + \sum_{i=1}^k \epsilon_i > 0$ we know that either there exists $i \in I_\delta$ so that $\delta_i > 0$, or there exists $i \in I_\epsilon$ so that $\epsilon_i > 0$. It is sufficient to show $x_{N-i} > 0$ and $y_{N-i} > 0$. However since $i \in I_\delta \cup I_\epsilon$ it follows that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|i$. Thus $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N-i$. Hence by our induction hypothesis $x_{N-i} > 0$ and $y_{N-i} > 0$. Thus $y_N > 0$. We have shown the second case.
Now assume it is not true that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$. Again the denominators are clearly non-zero and furthermore,

$$x_N = \frac{\sum_{i=1}^k \beta_i x_{N-i} + \sum_{i=1}^k \gamma_i y_{N-i}}{A + \sum_{j=1}^k B_j x_{N-j} + \sum_{j=1}^k C_j y_{N-j}} \leq \frac{\sum_{i=1}^k \beta_i x_{N-i} + \sum_{i=1}^k \gamma_i y_{N-i}}{A}.$$

Take $i \in I_\beta \cup I_\gamma$ it follows that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|i$. Hence by our assumption we have that for all $i \in I_\beta \cup I_\gamma$ it is not true that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N-i$. Indeed, assume $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N-i$ for some $i \in I_\beta \cup I_\gamma$, then $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$ contradicting our hypothesis. Hence by our induction hypothesis $x_{N-i} = 0$ and $y_{N-i} = 0$ for all $i \in I_\beta \cup I_\gamma$. So $\sum_{i=1}^k \beta_i x_{N-i} + \sum_{i=1}^k \gamma_i y_{N-i} = 0$. So $x_N = 0$ in this case. Thus we have shown the third case.
Again assume it is not true that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$. The denominators are clearly

non-zero and furthermore,

$$y_N = \frac{\sum_{i=1}^k \delta_i x_{N-i} + \sum_{i=1}^k \epsilon_i y_{N-i}}{q + \sum_{j=1}^k D_j x_{N-j} + \sum_{j=1}^k E_j y_{N-j}} \leq \frac{\sum_{i=1}^k \delta_i x_{N-i} + \sum_{i=1}^k \epsilon_i y_{N-i}}{q}.$$

Take $i \in I_\delta \cup I_\epsilon$ it follows that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|i$. Hence by our assumption we have that for all $i \in I_\delta \cup I_\epsilon$ it is not true that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N - i$. Indeed, assume $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N - i$ for some $i \in I_\delta \cup I_\epsilon$, then $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$ contradicting our hypothesis. Hence by our induction hypothesis $x_{N-i} = 0$ and $y_{N-i} = 0$ for all $i \in I_\delta \cup I_\epsilon$. So $\sum_{i=1}^k \delta_i x_{N-i} + \sum_{i=1}^k \epsilon_i y_{N-i} = 0$. So $y_N = 0$ in this case. Thus we have shown the fourth case.

Now we will make use of the prior result. Choose n such that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|n$. There does not exist $j \in I_B \cup I_C \cup I_D \cup I_E$ so that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|j$. Thus there does not exist $j \in I_B \cup I_C \cup I_D \cup I_E$ so that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|n - j$. Using this and the prior result, it follows that for n where $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|n$, $x_{n-j} = 0 = y_{n-j}$ for all $j \in I_B \cup I_C \cup I_D \cup I_E$. So for this choice of n we get

$$x_n = \frac{\sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A}, \tag{4}$$

$$y_n = \frac{\sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q}. \tag{5}$$

We now have two cases to consider. In case 1, $A < \sum_{i=1}^k \beta_i$, and $I_\delta \neq \emptyset$ so we use equation (4) and we get

$$x_n = \frac{\sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A} \geq \frac{\sum_{i=1}^k \beta_i x_{n-i}}{A} \geq \frac{\sum_{i=1}^k \beta_i}{A} \min_{i \in I_\beta}(x_{n-i}).$$

For convienience we now define $L = \gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)$. By our choice of n, we may write $m = \frac{n}{L} \in \mathbb{N}$. So our inequality reduces in this case to

$$x_{mL} \geq (\frac{\sum_{i=1}^k \beta_i}{A}) \min_{i \in I_\beta}(x_{mL-i})$$

$$\geq (\frac{\sum_{i=1}^k \beta_i}{A}) \min_{i \in \{1,\ldots,\lfloor \frac{k}{L} \rfloor\}}(x_{mL-iL}).$$

This is a difference inequality which holds for the subsequence $\{x_{mL}\}$ for $m \geq k$. We now rename this subsequence and apply the methods used in [4]. We set $z_m = x_{mL}$ for $m \in \mathbb{N}$. As we have just shown $\{z_m\}$ satisfies the following difference inequality,

$$z_m \geq (\frac{\sum_{i=1}^k \beta_i}{A}) \min_{i \in \{1,\ldots,\lfloor \frac{k}{L} \rfloor\}}(z_{m-i}), m \geq k.$$

Using the results of [4], particularly theorem 3, we have that for $m \geq k$,

$$\min(z_{m-1}, \ldots, z_{m-\lfloor \frac{k}{L} \rfloor}) \geq \min(u_{\lfloor \frac{m-k}{\lfloor \frac{k}{L} \rfloor} \rfloor}, \ldots, u_{m-k}).$$

Where $\{u_m\}_{m=0}^{\infty}$ is a solution of the difference equation,

$$u_m = (\frac{\sum_{i=1}^{k} \beta_i}{A})u_{m-1}, m \in \mathbb{N}. \tag{6}$$

With $u_0 = \min(z_{k-1}, \dots, z_{k-\lfloor \frac{k}{L} \rfloor})$.

Since we are in case 1, we know that $0 < A < \sum_{i=1}^{k} \beta_i$ and so every positive solution diverges to ∞ for the simple difference equation (6). Hence using the inequality we have obtained, $\{z_m\}_{m=1}^{\infty}$ diverges to ∞. Hence with given initial conditions, there is a subsequence of our solution $\{x_n\}_{n=1}^{\infty}$, namely $\{x_{mL}\}_{m=1}^{\infty}$, which diverges to ∞. Hence our solution $\{x_n\}_{n=1}^{\infty}$ is unbounded. Moreover since $\{x_{mL}\}_{m=1}^{\infty}$ diverges to ∞ and $I_\delta \neq \emptyset$, by equation (5), $\{y_{mL}\}_{m=1}^{\infty}$ diverges to ∞. So we have exhibited a solution in the case 1 where both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are unbounded. In case 2, $q < \sum_{i=1}^{k} \epsilon_i$, and $I_\gamma \neq \emptyset$ so we use equation (5) and we get

$$y_n = \frac{\sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q} \geq \frac{\sum_{i=1}^{k} \epsilon_i y_{n-i}}{q} \geq \frac{\sum_{i=1}^{k} \epsilon_i}{q} \min_{i \in I_\epsilon}(y_{n-i}).$$

For convienience we now define $L = \gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)$. By our choice of n, we may write $m = \frac{n}{L} \in \mathbb{N}$. So our inequality reduces in this case to

$$y_{mL} \geq (\frac{\sum_{i=1}^{k} \epsilon_i}{q}) \min_{i \in I_\epsilon}(y_{mL-i})$$

$$\geq (\frac{\sum_{i=1}^{k} \epsilon_i}{q}) \min_{i \in \{1,\dots,\lfloor \frac{k}{L} \rfloor\}}(y_{mL-iL}).$$

This is a difference inequality which holds for the subsequence $\{y_{mL}\}$ for $m \geq k$. We now rename this subsequence and apply the methods used in [4]. We set $w_m = y_{mL}$ for $m \in \mathbb{N}$. As we have just shown $\{w_m\}$ satisfies the following difference inequality,

$$w_m \geq (\frac{\sum_{i=1}^{k} \epsilon_i}{q}) \min_{i \in \{1,\dots,\lfloor \frac{k}{L} \rfloor\}}(w_{m-i}), m \geq k.$$

Using the results of [4], particularly theorem 3, we have that for $m \geq k$,

$$\min(w_{m-1}, \dots, w_{m-\lfloor \frac{k}{L} \rfloor}) \geq \min(v_{\lfloor \frac{m-k}{\lfloor \frac{k}{L} \rfloor} \rfloor}, \dots, v_{m-k}).$$

Where $\{v_m\}_{m=0}^{\infty}$ is a solution of the difference equation,

$$v_m = (\frac{\sum_{i=1}^{k} \epsilon_i}{q})v_{m-1}, m \in \mathbb{N}. \tag{7}$$

With $v_0 = \min(w_{k-1}, \dots, w_{k-\lfloor \frac{k}{L} \rfloor})$.

Since we are in case 2, we know that $0 < q < \sum_{i=1}^{k} \epsilon_i$ and so every positive solution diverges to ∞ for the simple difference equation (7). Hence using the inequality we have obtained, $\{w_m\}_{m=1}^{\infty}$ diverges to ∞. Hence with given initial conditions, there is a subsequence of our solution $\{y_n\}_{n=1}^{\infty}$, namely $\{y_{mL}\}_{m=1}^{\infty}$, which diverges to ∞. Hence our solution $\{y_n\}_{n=1}^{\infty}$ is unbounded. Moreover since $\{y_{mL}\}_{m=1}^{\infty}$ diverges to ∞ by equation

(4) and $I_\gamma \neq \emptyset$, $\{x_{mL}\}_{m=1}^{\infty}$ diverges to ∞. So we have exhibited a solution in the case 2 where both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are unbounded.

□

There is a very general idea taking place here. Look at a system of rational equations of the type presented here and look at the delays present in all of the numerators and all of the denominators. Does the greatest common divisor of all the delays in all the numerators divide some delay in one of the denominators? If the answer is no, we conjecture that a result similar to the one presented above can be shown for the system in question. The proof may be almost a duplicate of the above proof. We leave this proof to the determined reader.

4. Some Examples for Rational Systems in the Plane

Although these methods are intended to demonstrate unboundedness for systems of rational difference equations of order greater than one there are several examples of rational systems in the plane where these techniques apply. Here we present all first order rational systems in the plane where Theorem 1 applies.

Example 4. *Consider the system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_1 x_{n-1} + \gamma_1 y_{n-1}}{A + B_1 x_{n-1}}, n \in \mathbb{N},$$

$$y_n = \frac{p + \delta_1 x_{n-1} + \epsilon_1 y_{n-1}}{q + E_1 y_{n-1}}, n \in \mathbb{N},$$

with $\alpha, \beta_1, A, p, \epsilon_1, q \geq 0$, $\delta_1, \gamma_1, B_1, E_1 > 0$, and non-negative initial conditions. Assume that

(1) $\frac{\delta_1}{q+E_1} > 1$,
(2) $\frac{\gamma_1}{A+B_1} > 2$,
(3) $\frac{\alpha+1+\beta_1}{B_1} < 1$,
(4) $\frac{p+1+\epsilon_1}{E_1} < 1$.

then for some choice of initial conditions $\lim_{n\to\infty} x_{2n+1} = \infty$ *and* $\lim_{n\to\infty} y_{2n} = \infty$.

Proof. We apply Theorem 1. We let $a = 2$ and $b = 1$. We have the following

$$\begin{gathered} I_B^2 = \{1\} = \{1, \ldots, 2-1\} \\ I_E^2 = \{1\} = \{1, \ldots, 2-1\} \\ (I_\gamma \setminus I_C)^2 = \{1\} \\ (I_\delta \setminus I_D)^2 = \{1\} \\ (I_\beta \setminus I_B)^2 = \emptyset \\ (I_\epsilon \setminus I_E)^2 = \emptyset \\ 1 \notin \emptyset \end{gathered}$$

Thus Theorem 1 applies. □

Notice that in the above example the parameters $\alpha, \beta_1, A, p, \epsilon_1$, and q were allowed to be either positive or zero. Thus there are 64 rational systems in the plane for which Theorem 1 applies. Some of these cases have been covered by prior work, however the conjectures (23,23), (23,31), (23,34), (23,46), (31,31), (31,34), (31,46), (34,34), (34,46), and (46,46) in appendix 3 of [2] are covered by Example 4.

5. Conclusion

We have presented here several general results which prove the existence of unbounded solutions for systems of two rational difference equations of order greater than one. We feel that a good direction for further study would be to develop similar techniques which prove the existence of unbounded solutions for systems of more than two rational difference equations. We have given some limited guidance toward this goal in section 3. We would like to make reference to [1] and [2] for other work regarding systems of rational equations.

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