

# UNBOUNDEDNESS FOR SOME CLASSES OF RATIONAL DIFFERENCE EQUATIONS

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ABSTRACT. We study the rational difference equation

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, n \in \mathbb{N}.$$

Particularly, we show that for non-negative  $\alpha$  and  $C$ , whenever  $C\alpha = 0$  and  $C + \alpha > 0$ , unbounded solutions exist for some choice of non-negative initial conditions. Moreover, we study the rational difference equation

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, n \in \mathbb{N}.$$

Particularly, we show that whenever  $0 < \beta < \frac{1}{3}$  and  $\alpha \in [0, 1]$ , unbounded solutions exist for some choice of non-negative initial conditions.

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## 1. INTRODUCTION

In [1] Camouzis and Ladas devote a chapter to the study of unbounded solutions for the  $k^{th}$  order rational difference equation with non-negative parameters and non-negative initial conditions

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j}}, n \in \mathbb{N}.$$

In the introduction of said chapter, the authors of [1] pose five conjectures regarding the boundedness character of five different special cases of the third order rational difference equation. Particularly we are referring to the special cases #28, #44, #56, #70, and #120. These are the only remaining cases of third order for which the boundedness character has not been established.

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First we study special cases #56 and #120.

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, n \in \mathbb{N}. \quad (1)$$

Using a standard induction technique, we show that whenever  $0 < \beta < \frac{1}{3}$  and  $\alpha \in [0, 1]$ , unbounded solutions exist for some choice of non-negative initial conditions.

We then study special cases #44 and #28.

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, n \in \mathbb{N}. \quad (2)$$

We show that for non-negative  $\alpha$  and  $C$ , whenever  $C\alpha = 0$  and  $C + \alpha > 0$ , unbounded solutions exist for some choice of non-negative initial conditions. The proof is presented in two special cases. The case where  $\alpha > 0$  and the case where  $C > 0$ .

## 2. TODD'S EQUATION

Consider the third order rational difference equation.

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, n \in \mathbb{N}. \quad (3)$$

There have been significant results concerning the case where  $\beta = 1$ . In this case the equation is generally referred to by the cognomen "Todd's equation" and possesses the following invariant:

$$(\alpha + x_n + x_{n-1} + x_{n-2})\left(1 + \frac{1}{x_n}\right)\left(1 + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_{n-2}}\right) = \text{constant}.$$

For more information regarding Todd's equation see [4-6].

In the following theorem we show that whenever  $0 < \beta < \frac{1}{3}$  and  $\alpha \in [0, 1]$ , unbounded solutions exist for some choice of non-negative initial conditions. Notably this resolves two of the remaining five conjectures regarding the boundedness character of third order rational difference equations.

**Theorem 1.** *Consider the third order rational difference equation,*

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, n \in \mathbb{N}. \quad (4)$$

*Suppose  $0 < \beta < \frac{1}{3}$  and  $\alpha \in [0, 1]$ , then Equation (4) has unbounded solutions for some initial conditions.*

*Proof.* Choose initial conditions so that

$$\min(x_0, x_{-2}) > \max\left(\frac{1}{\beta}, \frac{x_{-1}}{\beta}\right).$$

We shall first prove by induction that for all  $j \in \mathbb{N}$ ,

$$\min(x_{2j}, x_{2j-2}) > \max\left(\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right). \quad (5)$$

The initial conditions provide the base case.

Assume the following holds for some  $j \in \mathbb{N}$ ,

$$\min(x_{2j-2}, x_{2j-4}) > \max\left(\frac{1}{\beta}, \frac{x_{2j-3}}{\beta}\right).$$

Since  $\beta x_{2j-2} > x_{2j-3}, x_{2j-2} > \frac{1}{\beta}$ ,  $\beta x_{2j-2} > 1 \geq \alpha$ , and  $x_{2j-4} > \frac{1}{\beta} > 3$  we see that

$$x_{2j-1} = \frac{\alpha + \beta x_{2j-2} + x_{2j-3}}{x_{2j-4}} < \frac{3\beta x_{2j-2}}{x_{2j-4}} < \beta x_{2j-2}.$$

Thus we have shown

$$x_{2j-2} > \max\left(\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right).$$

Since  $\beta x_{2j-4} > x_{2j-3}$  and  $0 < \beta < \frac{1}{3}$  we have

$$x_{2j} = \frac{\alpha + \beta x_{2j-1} + x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{\beta x_{2j-4}} > \frac{3x_{2j-2}}{x_{2j-4}} = \frac{3\beta x_{2j-2}}{\beta x_{2j-4}} > \frac{x_{2j-1}}{\beta}.$$

Also

$$x_{2j} > \frac{x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{\beta x_{2j-2}} = \frac{1}{\beta}.$$

Thus

$$\min(x_{2j}, x_{2j-2}) > \max\left(\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right).$$

This completes the induction.

Using Equation (5) we now prove that  $x_{8\eta} > \frac{x_{8\eta-8}}{9\beta^2}$  for all  $\eta \in \mathbb{N}$ .

$$\begin{aligned} x_{8\eta} &= \frac{\alpha + \beta x_{8\eta-1} + x_{8\eta-2}}{x_{8\eta-3}} > \frac{x_{8\eta-2}}{x_{8\eta-3}} = \left(\frac{x_{8\eta-6}}{\alpha + \beta x_{8\eta-4} + x_{8\eta-5}}\right) \left(\frac{\alpha + \beta x_{8\eta-3} + x_{8\eta-4}}{x_{8\eta-5}}\right) \\ &> \left(\frac{x_{8\eta-6}}{3\beta x_{8\eta-4}}\right) \left(\frac{x_{8\eta-4}}{x_{8\eta-5}}\right) = \frac{x_{8\eta-6}}{3\beta x_{8\eta-5}} = \frac{x_{8\eta-6} x_{8\eta-8}}{3\beta(\alpha + \beta x_{8\eta-6} + x_{8\eta-7})} > \frac{x_{8\eta-6} x_{8\eta-8}}{9\beta^2 x_{8\eta-6}} = \frac{x_{8\eta-8}}{9\beta^2}. \end{aligned}$$

Since  $0 < \beta < \frac{1}{3}$ ,  $9\beta^2 < 1$ . Thus we have a subsequence of our solution which diverges to  $\infty$ . Hence the solution is unbounded.  $\square$

### 3. SPECIAL CASE #44

We now study special case #44.

$$x_n = \frac{\alpha + x_{n-1}}{x_{n-3}}, n \in \mathbb{N}. \quad (6)$$

Particularly, we show that whenever  $\alpha > 0$ , Equation (6) has unbounded solutions for some initial conditions.

The following lemma provides a useful technique for constructing divergent subsequences of solutions for rational difference equations.

**Lemma 1.** *Let  $\{x_n\}_{n=1}^\infty$  be a sequence in  $[0, \infty)$ . Suppose that there exists  $D > 1$  and hypotheses  $H_1, \dots, H_k$  so that for all  $n \in \mathbb{N}$  there exists  $p_n \in \mathbb{N}$  so that the following holds. Whenever  $x_{n-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$ , then  $x_{n+p_n-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$  and  $x_{n+p_n-1} \geq Dx_{n-1}$ . Further assume that for some  $N \in \mathbb{N}$ ,  $x_{N-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$  and  $x_{N-1} > 0$ . Then  $\{x_n\}_{n=1}^\infty$  is unbounded. Particularly  $\{x_{z_m-1}\}_{m=1}^\infty$  is a subsequence of  $\{x_n\}_{n=1}^\infty$  which diverges to  $\infty$ , where  $z_m = z_{m-1} + p_{z_{m-1}}$  and  $z_0 = N$ .*

*Proof.* Let  $z_m = z_{m-1} + p_{z_{m-1}}$  and  $z_0 = N$ . Using induction, we prove that given  $m \in \mathbb{N}$  the following holds.  $x_{z_m-1} \geq D^m x_{N-1}$  and  $x_{z_m-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$ . By assumption,  $x_{N-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$  and  $x_{N-1} \geq D^0 x_{N-1}$ . This provides the base case. Assume  $x_{z_{m-1}-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$  and  $x_{z_{m-1}-1} \geq D^{m-1} x_{N-1}$ . Using our earlier assumption this implies that there exists  $p_{z_{m-1}}$  so that  $x_{z_{m-1}+p_{z_{m-1}}-i}$  satisfies  $H_i$  for all  $i \in \{1, \dots, k\}$  and  $x_{z_{m-1}+p_{z_{m-1}}-1} \geq Dx_{z_{m-1}-1} \geq (D)D^{m-1} x_{N-1} = D^m x_{N-1}$ .

So we have shown that  $x_{z_m-1} \geq D^m x_{N-1}$  for all  $m \in \mathbb{N}$ . Hence the subsequence  $\{x_{z_m-1}\}_{m=1}^\infty$  of  $\{x_n\}_{n=1}^\infty$  clearly diverges to  $\infty$  since  $D > 1$ .  $\square$

The above argument merely simplifies the following arguments by removing a somewhat onerous construction.

**Theorem 2.** *Consider the third order rational difference equation,*

$$x_n = \frac{\alpha + x_{n-1}}{x_{n-3}}, n \in \mathbb{N}. \quad (7)$$

*Suppose  $\alpha > 0$ , then Equation (7) has unbounded solutions for some initial conditions.*

*Proof.* We choose initial conditions so that

$$\begin{aligned} x_0 &> \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right), \\ x_{-1} &> \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right), \\ x_{-2} &> \frac{\alpha}{2}. \end{aligned}$$

We show that there exists  $D = \frac{4}{3}$  so that for all  $n \in \mathbb{N}$  there exists  $p_n \in \{7, 8\}$  so that the following holds.

Whenever

$$\begin{aligned} x_{n-1} &> \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right), \\ x_{n-2} &> \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right), \\ x_{n-3} &> \frac{\alpha}{2}. \end{aligned}$$

Then we have

$$x_{n+p_n-1} > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right),$$

$$\begin{aligned}
x_{n+p_n-2} &> \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right), \\
x_{n+p_n-3} &> \frac{\alpha}{2}, \\
x_{n+p_n-1} &\geq \left(\frac{4}{3}\right)x_{n-1}.
\end{aligned}$$

First assume

$$\begin{aligned}
x_{n-1} &> \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right), \\
x_{n-2} &> \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right), \\
x_{n-3} &> \frac{\alpha}{2}.
\end{aligned}$$

Since  $x_{n-1}, x_{n-2}, x_{n-3} > 0$  we may write  $\eta = \log_2(x_{n-1})$ ,  $\ell = \log_2(x_{n-2})$ , and  $\rho = \log_2(x_{n-3})$ . Hence  $2^\eta = x_{n-1}$ ,  $2^\ell = x_{n-2}$ , and  $2^\rho = x_{n-3}$ . We use such representations for ease of computations. First we see that

$$x_n = \frac{\alpha + x_{n-1}}{x_{n-3}} = \frac{\alpha}{x_{n-3}} + \frac{x_{n-1}}{x_{n-3}} = \frac{\alpha}{2^\rho} + 2^{\eta-\rho}. \quad (8)$$

$$x_{n+1} = \frac{\alpha}{x_{n-2}} + \frac{x_n}{x_{n-2}} = \frac{\alpha}{2^\ell} + \left(\frac{\alpha}{2^\rho} + 2^{\eta-\rho}\right) \frac{1}{2^\ell} = \frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}. \quad (9)$$

$$x_{n+2} = \frac{\alpha}{x_{n-1}} + \frac{x_{n+1}}{x_{n-1}} = \frac{\alpha}{2^\eta} + \left(\frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}\right) \left(\frac{1}{2^\eta}\right) = \frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\eta+\rho}}. \quad (10)$$

$$x_{n+3} = \frac{\alpha}{x_n} + \frac{x_{n+2}}{x_n} = \frac{\alpha 2^\rho}{\alpha + 2^\eta} + \left(\frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\eta+\rho}}\right) \left(\frac{2^\rho}{\alpha + 2^\eta}\right). \quad (11)$$

We will make use of these identities later. We prove the result in two cases. Let us first assume  $\ell + \rho \geq \eta$ . We show that if this inequality is satisfied for some  $n \in \mathbb{N}$ , then  $p_n = 7$ . First we prove that  $x_{n+p_n-3} = x_{n+4} > \frac{\alpha}{2}$ . Notice that

$$x_{n+4} = \frac{\alpha + x_{n+3}}{x_{n+1}} > \frac{\alpha}{x_{n+1}}.$$

From Equation (9) we see that

$$\frac{\alpha}{x_{n+1}} = \frac{\alpha}{\frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}} = \frac{\alpha}{\alpha 2^{-\ell} + \alpha 2^{-\ell-\rho} + 2^{\eta-\ell-\rho}}.$$

We now use the assumption  $\ell + \rho \geq \eta$ . This assumption implies that  $2^{\eta-\ell-\rho} \leq 2^0 = 1$ . Earlier we assumed that  $2^{-\rho} < \frac{2}{\alpha}$ . Moreover, from our assumptions,  $2^\ell > (\alpha+1)^2 2^{11} = (\alpha^2 + 2\alpha + 1) 2^{11}$  so  $2^{-\ell} < \frac{1}{\alpha 2^{12}}$  and  $2^{-\ell} < 2^{-11}$ . Using these inequalities we obtain the following.

$$\frac{\alpha}{\alpha 2^{-\ell} + \alpha 2^{-\ell-\rho} + 2^{\eta-\ell-\rho}} > \frac{\alpha}{\alpha \frac{1}{\alpha 2^{12}} + \alpha 2^{-11} \frac{2}{\alpha} + 1} = \frac{\alpha}{2^{-12} + 2^{-10} + 1} > \frac{\alpha}{2}.$$

So we have shown  $x_{n+p_n-3} = x_{n+4} > \frac{\alpha}{2}$ .

We now prove that  $x_{n+p_n-2} = x_{n+5} > \max(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11})$ . Notice that

$$x_{n+5} = \frac{\alpha + x_{n+4}}{x_{n+2}} > \frac{x_{n+4}}{x_{n+2}} > \frac{\alpha}{2x_{n+2}}.$$

Since  $\ell + \rho \geq \eta$ ,  $2^{\ell+\rho} \geq 2^\eta$ . Moreover as we have recently shown  $2^\ell > 2^{11}$ , similarly  $2^\eta > 2^{15}$ . So  $\ell > 11 > 0$  and  $\eta > 15 > 0$ . So from Equation (10),

$$x_{n+2} = \frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}} < \frac{\alpha}{2^\eta} + \frac{\alpha}{2^\eta} + \frac{\alpha}{2^\eta} + \frac{1}{2^\eta} = \frac{3\alpha + 1}{2^\eta}. \quad (12)$$

Hence,

$$x_{n+5} > \frac{\alpha}{2x_{n+2}} > \frac{\alpha}{2 \frac{3\alpha+1}{2^\eta}} = \frac{\alpha}{3\alpha+1} (2^{\eta-1}) > \left(\frac{1}{3\alpha+1}\right) \max\left(\frac{2^{14}}{\alpha^2}, (\alpha+1)^4 2^{14}\right).$$

So,

$$\begin{aligned} x_{n+5} &> \left(\frac{1}{3\alpha+1}\right) \max\left(\frac{2^{14}}{\alpha^2}, (\alpha+1)^4 2^{14}\right) \geq \frac{(\alpha+1)^4 2^{14}}{3\alpha+1} > \frac{(\alpha+1)^4 2^{14}}{3\alpha+3} \\ &= \frac{(\alpha+1)^3 2^{14}}{3} > (\alpha+1)^2 2^{11}. \end{aligned}$$

When  $\alpha \geq 1$ ,  $\frac{1}{\alpha^2} \leq 1 < (\alpha+1)^2$  so  $(\alpha+1)^2 2^{11} = \max(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11})$ . Thus the only remaining case is when  $\alpha < 1$ . In this case we have the following,

$$x_{n+5} > \left(\frac{1}{3\alpha+1}\right) \max\left(\frac{2^{14}}{\alpha^2}, (\alpha+1)^4 2^{14}\right) \geq \frac{2^{14}}{(3\alpha+1)\alpha^2} > \frac{2^{14}}{4\alpha^2} > \frac{2^{11}}{\alpha^2}.$$

So we have shown  $x_{n+p_n-2} = x_{n+5} > \max(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11})$ .

We now prove that  $x_{n+p_n-1} = x_{n+6} \geq (\frac{4}{3})x_{n-1} > \max(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha})$ . First assume  $\max((\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha}) \geq 2^{\eta-\rho}$ . Notice that

$$x_{n+6} = \frac{\alpha + x_{n+5}}{x_{n+3}} > \frac{x_{n+5}}{x_{n+3}} = \frac{\alpha + x_{n+4}}{x_{n+2}x_{n+3}} > \frac{x_{n+4}}{x_{n+2}x_{n+3}} = \frac{\alpha + x_{n+3}}{x_{n+2}x_{n+3}x_{n+1}} > \frac{1}{x_{n+2}x_{n+1}}.$$

We use Equation (9), our induction assumption, our assumption that  $\max((\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha}) \geq 2^{\eta-\rho}$ , and the fact that  $2^{-\rho} < \frac{2}{\alpha}$  to obtain,

$$x_{n+1} = \frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{2^{\eta-\rho}}{2^\ell} < \frac{\alpha}{2^\ell} + \frac{1}{2^{\ell-1}} + \frac{\max((\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha})}{\max(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11})}.$$

Notice that if  $\alpha \geq 1$ ,

$$\frac{\max((\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha})}{\max(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11})} \leq \frac{(\alpha+1)2^5}{(\alpha+1)^2 2^{11}} = \frac{1}{(\alpha+1)2^6} < \frac{1}{(\alpha+1)2^3}.$$

Also if  $\alpha < 1$ ,

$$\frac{\max((\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha})}{\max(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11})} \leq \frac{(\alpha+1)2^5}{\alpha(\frac{2^{11}}{\alpha^2})} = \frac{(\alpha+1)\alpha}{2^6} < \frac{(\alpha+1)^2}{2^6} < \frac{1}{(\alpha+1)2^3}.$$

So,

$$\begin{aligned} x_{n+1} &< \frac{\alpha}{2^\ell} + \frac{1}{2^{\ell-1}} + \frac{1}{(\alpha+1)2^3} < \frac{\alpha}{(\alpha+1)^2 2^{11}} + \frac{1}{(\alpha+1)^2 2^{10}} + \frac{1}{(\alpha+1)2^3} \\ &< \frac{1}{(\alpha+1)2^{11}} + \frac{1}{(\alpha+1)2^{10}} + \frac{1}{(\alpha+1)2^3} = \frac{1}{\alpha+1} (2^{-11} + 2^{-10} + 2^{-3}) < \frac{1}{4\alpha+4}. \end{aligned}$$

Now using the inequality we have just shown and Equation (12) we have,

$$x_{n+6} > \frac{1}{x_{n+1}x_{n+2}} > \frac{4\alpha+4}{3\alpha+1}(2^\eta) > \frac{4\alpha+4}{3\alpha+3}(2^\eta) = \left(\frac{4}{3}\right)x_{n-1}.$$

Thus we have shown  $x_{n+p_n-1} = x_{n+6} \geq \left(\frac{4}{3}\right)x_{n-1} > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right)$  when  $\max\left((\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha}\right) \geq 2^{\eta-\rho}$ . Now assume  $\max\left((\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha}\right) < 2^{\eta-\rho}$ . Using Equation (11) and Equation (12) we have the following,

$$x_{n+3} = \frac{\alpha 2^\rho}{\alpha+2^\eta} + (x_{n+2})\left(\frac{2^\rho}{\alpha+2^\eta}\right) < \frac{\alpha 2^\rho}{\alpha+2^\eta} + \left(\frac{3\alpha+1}{2^\eta}\right)\left(\frac{2^\rho}{\alpha+2^\eta}\right) < 2^{\rho-\eta}\left(\alpha + \frac{3\alpha+1}{2^\eta}\right).$$

So,

$$x_{n+6} > \frac{x_{n+5}}{x_{n+3}} > \frac{\alpha+x_{n+4}}{x_{n+3}x_{n+2}} > \frac{x_{n+4}}{x_{n+3}x_{n+2}} > \frac{\alpha}{x_{n+3}x_{n+2}x_{n+1}} > \frac{\alpha}{2^{\rho-\eta}\left(\alpha + \frac{3\alpha+1}{2^\eta}\right)x_{n+2}x_{n+1}}.$$

Since  $2^\eta > \frac{(\alpha+1)^4 2^{15}}{\alpha} > (\alpha+1)^3 2^{15} > 3\alpha+1$  we see that  $\frac{3\alpha+1}{2^\eta} < 1$  and using Equation (9) we get,

$$x_{n+6} > \frac{\alpha}{2^{\rho-\eta}(\alpha+1)x_{n+2}x_{n+1}} = \frac{\alpha}{2^{\rho-\eta}(\alpha+1)x_{n+2}\left(\frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}\right)}.$$

Distributing the  $2^{\rho-\eta}$  we have,

$$x_{n+6} > \frac{\alpha}{(\alpha+1)x_{n+2}\left(\frac{\alpha}{2^{\ell+\eta-\rho}} + \frac{\alpha}{2^{\ell+\eta}} + \frac{1}{2^\ell}\right)}. \quad (13)$$

Now let us assume  $\alpha \geq 1$  then we have,

$$x_{n+6} > \frac{\alpha}{(\alpha+1)x_{n+2}\left(\frac{\alpha}{2^{\ell+\eta-\rho}} + \frac{\alpha}{2^{\ell+\eta}} + \frac{\alpha}{2^\ell}\right)} = \frac{2^\ell}{(\alpha+1)x_{n+2}\left(\frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta} + 1\right)}.$$

In this case  $2^\ell > \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right) \geq (\alpha+1)^2 2^{11}$  so,

$$x_{n+6} > \frac{2^{11}(\alpha+1)}{x_{n+2}\left(1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta}\right)}.$$

Since we assumed that  $\max\left((\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha}\right) < 2^{\eta-\rho}$  we have that  $2^{\eta-\rho} > 2^5 > 1$ . Furthermore we know from earlier that  $2^\eta > 2^{15} > 1$ . Using this information we obtain,

$$x_{n+6} > \frac{2^{11}(\alpha+1)}{x_{n+2}\left(1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta}\right)} > \frac{2^{11}(\alpha+1)}{3x_{n+2}}.$$

Now we use Equation (12) and we obtain,

$$x_{n+6} > \frac{2^{11}(\alpha+1)}{3(3\alpha+1)}(x_{n-1}) > \frac{2^{11}(\alpha+1)}{3(3\alpha+3)}(x_{n-1}) = \left(\frac{2^{11}}{9}\right)x_{n-1} > \left(\frac{4}{3}\right)x_{n-1}.$$

We now prove the case when  $\alpha < 1$ , here we continue from Equation (13) with the following,

$$x_{n+6} > \frac{\alpha}{(\alpha + 1)x_{n+2}(\frac{1}{2^{\ell+\eta-\rho}} + \frac{1}{2^{\ell+\eta}} + \frac{1}{2^\ell})} = \frac{\alpha 2^\ell}{(\alpha + 1)x_{n+2}(\frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta} + 1)}.$$

In this case  $2^\ell > \max(\frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11}) \geq \frac{2^{11}}{\alpha}$  so we have,

$$x_{n+6} > \frac{2^{11}}{(\alpha + 1)x_{n+2}(1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta})}.$$

Since we assumed that  $\max((\alpha + 1)2^5, \frac{(\alpha+1)2^5}{\alpha}) < 2^{\eta-\rho}$  we have that  $2^{\eta-\rho} > 2^5 > 1$ . Furthermore we know from earlier that  $2^\eta > 2^{15} > 1$ . Using this information we obtain,

$$x_{n+6} > \frac{2^{11}}{(\alpha + 1)x_{n+2}(1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^\eta})} > \frac{2^{11}}{3(\alpha + 1)x_{n+2}}.$$

Now we use Equation (12) and the assumption  $\alpha < 1$  to obtain,

$$x_{n+6} > \frac{2^{11}}{3(3\alpha + 1)(\alpha + 1)}(x_{n-1}) > \frac{2^{11}}{24}(x_{n-1}) > \left(\frac{4}{3}\right)x_{n-1}.$$

Thus we have shown  $x_{n+p_n-1} = x_{n+6} \geq \left(\frac{4}{3}\right)x_{n-1} > \max(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)4 \cdot 2^{15}}{\alpha})$  when

$\max((\alpha + 1)2^5, \frac{(\alpha+1)2^5}{\alpha}) < 2^{\eta-\rho}$ . Therefore we have finished the case where  $\ell + \rho \geq \eta$ .

We now consider the case  $\ell + \rho < \eta$ . We show that if this inequality is satisfied for some  $n \in \mathbb{N}$  then  $p_n = 8$ . First we prove that  $x_{n+p_n-3} = x_{n+5} > \frac{\alpha}{2}$ . Notice that since our assumptions have changed, Equations (12) and (13) no longer hold. We will now make a new analogue for Equation (12), namely Equation (14). Since  $\ell + \rho < \eta$ ,  $2^{\ell+\rho} < 2^\eta$ . Moreover, since  $2^\ell > 2^{11}$  and  $2^\eta > 2^{15}$ ,  $\ell > 0$  and  $\eta > 0$ . So from Equation (10),

$$x_{n+2} = \frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}} < \frac{\alpha}{2^{\ell+\rho}} + \frac{\alpha}{2^{\ell+\rho}} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho}} = \frac{3\alpha + 1}{2^{\ell+\rho}}. \quad (14)$$

So we have,

$$x_{n+5} = \frac{\alpha + x_{n+4}}{x_{n+2}} > \frac{\alpha}{x_{n+2}} > \frac{\alpha 2^{\ell+\rho}}{3\alpha + 1}.$$

Notice that,

$$\begin{aligned} \frac{\alpha 2^\ell}{3\alpha + 1} &> \max\left(\frac{\alpha 2^{11}}{(3\alpha + 1)\alpha^2}, \frac{\alpha(\alpha + 1)^2 2^{11}}{(3\alpha + 1)}\right) \geq \max\left(\frac{2^{11}}{(3\alpha + 1)\alpha}, \frac{\alpha(\alpha + 1)^2 2^{11}}{(3\alpha + 3)}\right) \\ &= \max\left(\frac{2^{11}}{(3\alpha + 1)\alpha}, \frac{\alpha(\alpha + 1)2^{11}}{3}\right) > 1. \end{aligned}$$

So  $x_{n+5} > 2^\rho > \frac{\alpha}{2}$ .

We now prove that  $x_{n+p_n-2} = x_{n+6} > \max(\frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11})$ . Notice that from Equation (11) we have,

$$x_{n+3} = \frac{\alpha 2^\rho}{\alpha + 2^\eta} + (x_{n+2})\left(\frac{2^\rho}{\alpha + 2^\eta}\right) < 2^{\rho-\eta}(\alpha + x_{n+2}). \quad (15)$$

Since  $x_{n+5} > 2^\rho$  we get,

$$x_{n+6} > \frac{x_{n+5}}{x_{n+3}} > 2^{\eta-\rho} \frac{x_{n+5}}{\alpha + x_{n+2}} > \frac{2^\eta}{\alpha + x_{n+2}}. \quad (16)$$

We assume  $\frac{1}{\alpha} \leq \alpha + 1$  and we use Equation (14). We know that  $2^\rho > \frac{\alpha}{2}$  and  $2^\ell > (\alpha + 1)^2 2^{11} \geq \frac{(\alpha+1)2^{11}}{\alpha}$ . So,

$$x_{n+2} < \frac{3\alpha + 1}{2^{\ell+\rho}} < \frac{3\alpha + 1}{\frac{(\alpha+1)2^{11}}{\alpha} \left(\frac{\alpha}{2}\right)} = \frac{3\alpha + 1}{(\alpha + 1)2^{10}} < \frac{3\alpha + 3}{(\alpha + 1)2^{10}} < 2^{-8}.$$

So, since  $\frac{1}{\alpha} \leq \alpha + 1$ ,

$$\begin{aligned} x_{n+6} &> \frac{2^\eta}{\alpha + x_{n+2}} > \frac{2^\eta}{\alpha + 2^{-8}} > \max\left(\frac{2^{15}}{(\alpha + 2^{-8})\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{(\alpha + 2^{-8})\alpha}\right) \\ &\geq \frac{(\alpha + 1)^4 2^{15}}{(\alpha + 2^{-8})\alpha} > \frac{(\alpha + 1)^4 2^{15}}{(\alpha + 1)^2} = (\alpha + 1)^2 2^{15} > (\alpha + 1)^2 2^{11} \geq \frac{2^{11}}{\alpha^2}. \end{aligned}$$

We now assume  $\frac{1}{\alpha} > \alpha + 1$  and we use Equation (14). We know that  $2^\rho > \frac{\alpha}{2}$  and  $2^\ell > \frac{2^{11}}{\alpha^2} \geq \frac{(\alpha+1)2^{11}}{\alpha}$ . So,

$$x_{n+2} < \frac{3\alpha + 1}{2^{\ell+\rho}} < \frac{3\alpha + 1}{\frac{(\alpha+1)2^{11}}{\alpha} \left(\frac{\alpha}{2}\right)} = \frac{3\alpha + 1}{(\alpha + 1)2^{10}} < \frac{3\alpha + 3}{(\alpha + 1)2^{10}} < 2^{-8}.$$

We now use Equation (16) and our assumption  $\frac{1}{\alpha} > \alpha + 1$  to obtain the following.

$$\begin{aligned} x_{n+6} &> \frac{2^\eta}{\alpha + x_{n+2}} > \frac{2^\eta}{\alpha + 2^{-8}} > \max\left(\frac{2^{15}}{(\alpha + 2^{-8})\alpha^3}, \frac{(\alpha + 1)^4 2^{15}}{(\alpha + 2^{-8})\alpha}\right) \\ &\geq \frac{2^{15}}{(\alpha + 2^{-8})\alpha^3} > \frac{(\alpha + 1)2^{15}}{(\alpha + 1)\alpha^2} = \frac{2^{15}}{\alpha^2} > \frac{2^{11}}{\alpha^2} > (\alpha + 1)^2 2^{11}. \end{aligned}$$

Thus we have shown that  $x_{n+p_n-2} = x_{n+6} > \max\left(\frac{2^{11}}{\alpha^2}, (\alpha + 1)^2 2^{11}\right)$ .

Now we prove  $x_{n+p_n-1} = x_{n+7} \geq \left(\frac{4}{3}\right)x_{n-1} > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right)$ . Notice that,

$$x_{n+7} = \frac{\alpha + x_{n+6}}{x_{n+4}} > \frac{x_{n+6}}{x_{n+4}} = \frac{\alpha + x_{n+5}}{x_{n+4}x_{n+3}} > \frac{x_{n+5}}{x_{n+4}x_{n+3}} = \frac{\alpha + x_{n+4}}{x_{n+2}x_{n+3}x_{n+4}} > \frac{1}{x_{n+2}x_{n+3}}.$$

Using Equations (14) and (15) we have,

$$x_{n+7} > \frac{1}{x_{n+2}x_{n+3}} > \frac{2^{\ell+\rho}}{(3\alpha + 1)x_{n+3}} > \frac{2^{\ell+\eta}}{(3\alpha + 1)(\alpha + x_{n+2})}.$$

Earlier we demonstrated that  $x_{n+2} < 2^{-8}$ . Furthermore we have assumed that  $2^\ell > (\alpha + 1)^2 2^{11}$ . Thus,

$$x_{n+7} > \frac{2^{\ell+\eta}}{(3\alpha + 1)(\alpha + x_{n+2})} > \frac{2^{\ell+\eta}}{(3\alpha + 3)(\alpha + 1)} > \frac{(\alpha + 1)^2 2^{11} 2^\eta}{(3\alpha + 3)(\alpha + 1)} = \left(\frac{2^{11}}{3}\right)x_{n-1} > \left(\frac{4}{3}\right)x_{n-1}.$$

Hence  $x_{n+p_n-1} = x_{n+7} \geq \left(\frac{4}{3}\right)x_{n-1} > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right)$ . Now we apply Lemma 1 and then the proof is done.  $\square$

## 4. SPECIAL CASE #28

We now study special case #28.

$$x_n = \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}}, n \in \mathbb{N}. \quad (17)$$

Particularly, we show that whenever  $C > 0$ , Equation (17) has unbounded solutions for some initial conditions.

**Theorem 3.** *Consider the third order rational difference equation,*

$$x_n = \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}}, n \in \mathbb{N}. \quad (18)$$

*Suppose  $C > 0$ , then Equation (18) has unbounded solutions for some initial conditions.*

*Proof.* We choose initial conditions so that

$$x_0 > \max(1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)),$$

$$x_{-1} > \max(10, \frac{10}{C}, \frac{1}{C^3}).$$

We show that there exists  $D = 2$  so that for all  $n \in \mathbb{N}$  there exists  $p_n \in \{7, 8\}$  so that the following holds.

Whenever

$$x_{n-1} > \max(1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)),$$

$$x_{n-2} > \max(10, \frac{10}{C}, \frac{1}{C^3}).$$

Then we have

$$x_{n+p_n-1} > \max(1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)),$$

$$x_{n+p_n-2} > \max(10, \frac{10}{C}, \frac{1}{C^3}),$$

$$x_{n+p_n-1} \geq 2x_{n-1}.$$

First assume

$$x_{n-1} > \max(1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)),$$

$$x_{n-2} > \max(10, \frac{10}{C}, \frac{1}{C^3}).$$

Using algebra we immediately obtain the following.

$$x_n = \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}} < \frac{x_{n-1}}{Cx_{n-2}} < \frac{x_{n-1}}{10}.$$

$$x_{n+1} = \frac{x_n}{Cx_{n-1} + x_{n-2}} < \frac{x_n}{x_{n-2}} < \frac{x_n}{10}.$$

$$\begin{aligned}
 x_{n+1} &= \frac{x_n}{Cx_{n-1} + x_{n-2}} < \frac{x_n}{Cx_{n-1}} < \frac{x_n}{1000C}. \\
 x_n &= \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}} < \frac{x_{n-1}}{Cx_{n-2}} < \frac{x_{n-1}}{10C}. \\
 x_{n+2} &= \frac{x_{n+1}}{Cx_n + x_{n-1}} < \frac{x_{n+1}}{x_{n-1}} < \frac{x_{n+1}}{100C}. \\
 x_{n+2} &= \frac{x_{n+1}}{Cx_n + x_{n-1}} < \frac{x_{n+1}}{x_{n-1}} < \frac{x_{n+1}}{1000}.
 \end{aligned}$$

So we get the following inequalities,

$$100000x_{n+2} < 100x_{n+1} < 10x_n < x_{n-1}. \quad (19)$$

$$1000000C^3x_{n+2} < 10000C^2x_{n+1} < 10Cx_n < x_{n-1}. \quad (20)$$

Using Equation (20) we get the following,

$$\frac{x_{n+2}}{x_{n+3}} = Cx_{n+1} + x_n < 2x_n. \quad (21)$$

We use Equations (19) and (20) to get,

$$\begin{aligned}
 Cx_{n+3} + x_{n+2} &= x_{n+2}\left(1 + \frac{C}{Cx_{n+1} + x_n}\right) < x_{n+2}\left(1 + \frac{1}{x_{n+1}}\right) < x_{n+2}\left(1 + \frac{1}{1000x_{n+2}}\right) \\
 &= x_{n+2} + \frac{1}{1000} = \frac{x_n}{(Cx_n + x_{n-1})(Cx_{n-1} + x_{n-2})} + \frac{1}{1000} < \frac{1}{Cx_{n-2}} + \frac{1}{1000} < \frac{1}{10} + \frac{1}{1000} < 1.
 \end{aligned}$$

In short,

$$Cx_{n+3} + x_{n+2} < 1. \quad (22)$$

Using Equations (20) and (21) we have,

$$\begin{aligned}
 x_{n+5} &= \frac{x_{n+4}}{Cx_{n+3} + x_{n+2}} = \frac{1}{Cx_{n+2} + x_{n+1}} \left(\frac{x_{n+3}}{Cx_{n+3} + x_{n+2}}\right). \\
 x_{n+5} &= \frac{1}{Cx_{n+2} + x_{n+1}} \left(\frac{1}{C + \frac{x_{n+2}}{x_{n+3}}}\right) > \frac{1}{2x_{n+1}} \left(\frac{1}{C + 2x_n}\right).
 \end{aligned} \quad (23)$$

Furthermore we have the following,

$$x_{n+4} = \frac{x_{n+3}}{Cx_{n+2} + x_{n+1}} < \frac{x_{n+3}}{x_{n+1}} = \frac{1}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} < \frac{1}{x_{n-1}x_n}. \quad (24)$$

Using Equations (23) and (24) we obtain the following.

$$\begin{aligned}
 x_{n+6} &= \frac{x_{n+5}}{Cx_{n+4} + x_{n+3}} > \left(\frac{1}{2x_{n+1}}\right) \left(\frac{1}{C + 2x_n}\right) \left(\frac{1}{\frac{C}{x_{n-1}x_n} + x_{n+3}}\right) \\
 &= \left(\frac{x_{n-1}}{2x_{n+1}}\right) \left(\frac{1}{C + 2x_n}\right) \left(\frac{1}{\frac{C}{x_n} + x_{n-1}x_{n+3}}\right) = \left(\frac{Cx_{n-1} + x_{n-2}}{2C + 4x_n}\right) \left(\frac{Cx_{n-2} + x_{n-3}}{\frac{C}{x_n} + x_{n-1}x_{n+3}}\right) \\
 &= \left(\frac{(Cx_{n-1} + x_{n-2})^2}{2C + 4x_n}\right) \left(\frac{Cx_{n-2} + x_{n-3}}{\frac{C}{x_{n+1}} + x_{n-1}x_{n+3}(Cx_{n-1} + x_{n-2})}\right).
 \end{aligned}$$

Using Equations (19) and (20) we see

$$\begin{aligned} x_{n-1}x_{n+3}(Cx_{n-1} + x_{n-2}) &= \frac{x_{n-1}x_{n+2}(Cx_{n-1} + x_{n-2})}{Cx_{n+1} + x_n} = \frac{x_{n-1}x_{n+1}(Cx_{n-1} + x_{n-2})}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} \\ &= \frac{x_{n-1}x_n}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} < 1 < C^3x_{n-2} < \frac{C(Cx_{n-1} + x_{n-2})(Cx_{n-2} + x_{n-3})}{x_{n-1}} = \frac{C}{x_{n+1}}. \end{aligned}$$

Using this in the prior inequality we get,

$$x_{n+6} > \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{4C^2 + 8Cx_n}. \quad (25)$$

Using this fact we have the following,

$$\begin{aligned} x_{n+7} &= \frac{x_{n+6}}{Cx_{n+5} + x_{n+4}} = \frac{x_{n+6}}{\left(\frac{Cx_{n+4}}{Cx_{n+3} + x_{n+2}}\right) + x_{n+4}} = \left(\frac{x_{n+6}}{x_{n+4}}\right)\left(\frac{1}{1 + \frac{C}{Cx_{n+3} + x_{n+2}}}\right) \\ &= \left(\frac{x_{n+4}}{Cx_{n+3} + x_{n+2}}\right)\left(\frac{1}{Cx_{n+4} + x_{n+3}}\right)\left(\frac{1}{x_{n+4}}\right)\left(\frac{1}{1 + \frac{C}{Cx_{n+3} + x_{n+2}}}\right) \\ &= \left(\frac{1}{C + Cx_{n+3} + x_{n+2}}\right)\left(\frac{1}{Cx_{n+4} + x_{n+3}}\right) > \left(\frac{1}{Cx_{n+4} + x_{n+3}}\right)\left(\frac{1}{C + 1}\right). \end{aligned}$$

We now use Equations (19) and (20) to show the following,

$$\begin{aligned} x_{n+3} &= \frac{x_{n+2}}{Cx_{n+1} + x_n} = \frac{x_{n+1}}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} \\ &< \frac{x_{n+1}}{(Cx_{n+1} + 500Cx_{n+1} + 5x_{n+1})(Cx_n + x_{n-1})} < \frac{1}{(501C + 5)x_{n-1}} < \frac{1}{(4C + 4)x_{n-1}}. \end{aligned}$$

We now use this fact and Equation (24) to obtain the following,

$$x_{n+7} > \left(\frac{1}{\frac{C}{x_n x_{n-1}} + \frac{1}{x_{n-1}(4C+4)}}\right)\left(\frac{1}{C+1}\right) = \left(\frac{x_{n-1}}{\frac{C}{x_n} + \frac{1}{4(C+1)}}\right)\left(\frac{1}{C+1}\right). \quad (26)$$

Suppose  $x_n \leq 4C(C+1)$  we will show that in this case  $p_n = 7$ . Using Equation (23) we have,

$$\begin{aligned} x_{n+p_n-2} &= x_{n+5} > \frac{1}{2x_{n+1}}\left(\frac{1}{C + 2x_n}\right) = \\ &\frac{Cx_{n-1} + x_{n-2}}{2x_n}\left(\frac{1}{C + 2x_n}\right) > \frac{Cx_{n-1}}{2x_n(C + 2x_n)} > \frac{Cx_{n-1}}{8C(C+1)(C + 8C(C+1))} \\ &= \frac{x_{n-1}}{8(C+1)(C + 8C(C+1))} > \frac{x_{n-1}}{100(C+1)^3} > \max\left(10, \frac{10}{C}, \frac{1}{C^3}\right). \end{aligned}$$

Also using Equation (25) we have,

$$\begin{aligned} x_{n+p_n-1} &= x_{n+6} > \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{4C^2 + 8Cx_n} > \frac{Cx_{n-1}^2}{4C^2 + 8C(4C(C+1))} \\ &= \frac{x_{n-1}^2}{4C + 8(4C(C+1))} > \frac{x_{n-1}^2}{50(C^2 + C)} > 2x_{n-1}. \end{aligned}$$

Now suppose  $x_n > 4C(C + 1)$  we will show that in this case  $p_n = 8$ . Using Equation (25) we have,

$$\begin{aligned} x_{n+p_n-2} = x_{n+6} &> \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{4C^2 + 8Cx_n} > \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{9Cx_n} \\ &= \frac{(Cx_{n-1} + x_{n-2})(Cx_{n-2} + x_{n-3})}{9C} > x_{n-2} > \max\left(10, \frac{10}{C}, \frac{1}{C^3}\right). \end{aligned}$$

Also using Equation (26) we have,

$$x_{n+p_n-1} = x_{n+7} > \left(\frac{x_{n-1}}{\frac{C}{x_n} + \frac{1}{4(C+1)}}\right)\left(\frac{1}{C+1}\right) > \left(\frac{x_{n-1}}{\frac{C}{4C(C+1)} + \frac{1}{4(C+1)}}\right)\left(\frac{1}{C+1}\right) = 2x_{n-1}.$$

Hence after application of Lemma 1 the proof is complete.  $\square$

## 5. CONCLUSION

Theorem 1 establishes the boundedness character of special cases #56 and #120. Theorems 2 and 3 establish the boundedness character of the special cases #44 and #28 respectively. There remains only one special case of third order for which the boundedness character has not yet been established. This is special case #70. I will restate the conjecture regarding equation #70 for emphasis.

**Conjecture 1.** *Consider the third order rational difference equation,*

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, n \in \mathbb{N}. \quad (27)$$

*Equation (29) has unbounded solutions with  $\alpha, C > 0$  for some initial conditions.*

For more on boundedness character see [1-3].

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