ON PERIODIC TRICHOTOMIES

FRANK J. PALLADINO

Abstract. We study the k^{th} order rational difference equation

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}.$$

Particularly we observe trichotomy behavior which depends on a comparison between A and $\sum_{i=1}^{k} \beta_i$. We develop several new trichotomy results and resolve several conjectures regarding the boundedness character of the k^{th} order rational difference equation.

fpalladino@math.sunysb.edu

1. Introduction

In [2] Camouzis and Ladas devote a chapter to the study of periodic trichotomy results for the k^{th} order rational difference equation with non-negative parameters and non-negative initial conditions

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, n \in \mathbb{N}. \tag{1}$$

We find the results detailed in section 4.4 of [2] to be of particular interest. In this section the authors study the behavior in a special case of the general rational difference equation namely,

$$x_n = \frac{x_{n-3}}{A + Bx_{n-1} + Cx_{n-2}}, n \in \mathbb{N}. \tag{2}$$

They consider the case in which the initial conditions are non-negative and impose the restrictions $A, B, C \in [0, \infty)$ and $B + C > 0$. Under these assumptions the authors prove that the solutions of Equation (2) exhibit period-three trichotomy behavior. This trichotomy depends on the paramater A. Particularly the behavior is as follows.
When $A > 1$ every solution of Equation (2) converges to zero.
When $A = 1$ every solution of Equation (2) converges to a periodic solution of period 3.
When $0 \leq A < 1$, Equation (2) has unbounded solutions for some initial conditions.
Observe that upon considering this case as part of the larger framework of the general rational difference equation this trichotomy condition is restated as follows.

Date: June 5, 2008.
[0]**Keywords: difference equation, periodic convergence, global stability, trichotomy character.**
AMS Subject Classification: 39A10,39A11

When $A > \sum_{i=1}^{k} \beta_i$ every solution of Equation (2) converges to zero. When $A = \sum_{i=1}^{k} \beta_i$ every solution of Equation (2) converges to a periodic solution of period 3. When $0 \leq A < \sum_{i=1}^{k} \beta_i$, Equation (2) has unbounded solutions for some initial conditions.
Indeed, such a result holds on Equation (2) since $\sum_{i=1}^{3} \beta_i = 0 + 0 + 1 = 1$. Thus, in the case of Equation (2), such a statement is merely a rephrasing of the original result. This begets a very natural question; What conditions must we impose on Equation (1) to obtain a trichotomy result with the characteristics seen above? We explore the question herein.

2. The Question

Let us first clearly state our goal. We seek conditions which, when applied to Equation (1), guarantee the following trichotomy behavior.

i. When $A > \sum_{i=1}^{k} \beta_i$ every solution converges to the unique equilibrium.
ii. When $A = \sum_{i=1}^{k} \beta_i$ every solution converges to a periodic solution of not necessarily prime period p. So that this is a trichotomy and not a dichotomy we require p>1.
iii. When $0 \leq A < \sum_{i=1}^{k} \beta_i$ unbounded solutions exist for some initial conditions.

We see immediately that if $\sum_{i=1}^{k} \beta_i = 0$ then the condition $0 \leq A < \sum_{i=1}^{k} \beta_i$ is never satisfied. Hence in the sequel we assume $\sum_{i=1}^{k} \beta_i > 0$.
A result has recently appeared in the literature which serves to narrow our focus, namely [12]. The result in [12] complements the results in [2,4,5,10,13]. The combination of these results allows the behavior of solutions to Equation (1) to be explicitly described whenever $A \geq \sum_{i=1}^{k} \beta_i$. A concise summary of this behavior is contained in [12]. These results are vital to this paper so we rephrase the summary in the case $\sum_{i=1}^{k} \beta_i > 0$. Before doing so let us first introduce some notation. Let us define the following sets of indices :
$I_\beta = \{i \in \{1, 2, \ldots, k\} | \beta_i > 0\}$ and $I_B = \{j \in \{1, 2, \ldots, k\} | B_j > 0\}$.
These sets are used extensively in [12] when referring to the k^{th} order rational difference equation. Similarly we shall make extensive use of this notation.
The summary from [12] is as follows.
If $A > \sum_{i=1}^{k} \beta_i$, then the unique equilibrium of Equation (1) is globally asymptotically stable.
Whenever $A = \sum_{i=1}^{k} \beta_i$, and $\alpha = 0$, the following holds.

(1) If $gcd(I_\beta)|j$ for some $j \in I_B$ then the unique equilibrium of Equation (1) is globally asymptotically stable.
(2) If there does not exist $j \in I_B$ so that $gcd(I_\beta)|j$ then every solution of Equation (1) converges to a periodic solution of period $gcd(I_\beta)$.

Suppose $A = \sum_{i=1}^{k} \beta_i$, $\alpha > 0$, and $gcd(I_\beta \cup I_B) = 1$, then the following holds.

(1) If i is even for all $i \in I_\beta$ and j is odd for all $j \in I_B$, then every solution of Equation (1) converges to a periodic solution of period 2.
(2) If not, then every solution of Equation (1) converges to the unique equilibrium.

If $A = \sum_{i=1}^{k} \beta_i$, $\alpha > 0$, and $gcd(I_\beta \cup I_B) \neq 1$ then we must make a change of variables and apply the prior result.
The sole condition that $A > \sum_{i=1}^{k} \beta_i$ is sufficient to prove that every solution converges to the unique equilibrium, see [2,10]. Hence the result (i) is always true. The condition $A = \sum_{i=1}^{k} \beta_i$ is sufficient to ensure that every solution converges to a periodic solution. However, to guarantee trichotomy behavior, there must exist at least one solution which converges to a periodic solution of prime period $p > 1$. We see from our summary that it is necessary to assume one of the following.

I. We assume $\alpha > 0$, $\sum_{i=1}^{k} \beta_i > 0$, $gcd(I_\beta \cup I_B) = 1$, i is even for all $i \in I_\beta$, and j is odd for all $j \in I_B$.
II. We assume $\alpha > 0$, $\sum_{i=1}^{k} \beta_i > 0$, $gcd(I_\beta \cup I_B) \neq 1$ and after applying a change of variables i is even for all $i \in I_\beta$ and j is odd for all $j \in I_B$.
III. We assume $\alpha = 0$, $\sum_{i=1}^{k} \beta_i > 0$, and that there does not exist $j \in I_B$ so that $gcd(I_\beta)|j$.

Indeed, if $A = \sum_{i=1}^{k} \beta_i$, $\sum_{i=1}^{k} \beta_i > 0$, and none of the conditions (I), (II), or (III) hold then every solution converges to the unique equilibrium as was shown in [2,12,13]. Moreover, we shall show that this is also sufficient. In other words we shall exhibit a periodic solution of prime period $p > 1$ whenever one of the conditions (I), (II), or (III) hold. This is fairly straightforward and was shown in [2,12,13]. Hence in the case with $\sum_{i=1}^{k} \beta_i > 0$ it is both necessary and sufficient that one of the conditions (I), (II), or (III) hold in order for the result (ii) to be true.
We believe that whenever $0 \leq A < \sum_{i=1}^{k} \beta_i$ and one of the conditions (I), (II), or (III) hold unbounded solutions exist for some choice of initial conditions. In this way we believe no restriction beyond that imposed by the assumption $\sum_{i=1}^{k} \beta_i > 0$ and one of the conditions (I), (II), or (III) is required to ensure trichotomy behavior. In Theorem 1 we reference the work of [14] to show that whenever $0 \leq A < \sum_{i=1}^{k} \beta_i$ and one of the conditions (I) or (II) hold unbounded solutions exist for some choice of initial conditions. In Theorem 2 we show that whenever $0 < A < \sum_{i=1}^{k} \beta_i$ and condition (III) holds unbounded solutions exist for some choice of initial conditions. This leaves the following conjecture.

Conjecture 1. *Consider the k^{th} order rational difference equation,*

$$x_n = \frac{\sum_{i=1}^{k} \beta_i x_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j}}, n \in \mathbb{N}. \tag{3}$$

Assume non-negative parameters and non-negative initial conditions so that the denominator is non-vanishing. Further assume that $\sum_{i=1}^{k} \beta_i > 0$ and that there does not exist $j \in I_B$ so that $gcd(I_\beta)|j$. Then unbounded solutions of Equation (3) exist for some initial conditions.

We provide partial results for Conjecture 1 in Section 5. In Section 6 we prove Conjecture 1 in its entirety for cases of sufficiently small order. We conclude with emphasis on what remains of Conjecture 1 and possible directions for further work.

3. Prior Results

In [14] the authors, inspired by the results in [3] and [8], prove a general period 2 trichotomy result. It so happens that we may extend this result using the work in [2,10,12,13]. This is entirely due to the condition given by the authors in [14] that $t = i - sr$. This is an equation acting on the indices of Equation (1) which must be satisfied for their result to hold. We will make no further use of this restriction here and consequently advise the reader to refer to [14] for an explanation regarding the meaning of this equation.

It is our desire to alleviate this restriction and establish the trichotomy result in its full generality. Notice that the sole condition that $A > \sum_{i=1}^{k} \beta_i$ is sufficient to prove that every solution converges to the unique equilibrium, see [2,10]. Furthermore we know from [2,12,13] the following two facts. If $gcd(I_\beta \cup I_B) = 1$, i is even for all $i \in I_\beta$, j is odd for all $j \in I_B$, and $A = \sum_{i=1}^{k} \beta_i$ then every solution converges to a periodic solution of not necessarily prime period 2. If $gcd(I_\beta \cup I_B) \neq 1$ and after applying a change of variables i is even for all $i \in I_\beta$,j is odd for all $j \in I_B$, and $A = \sum_{i=1}^{k} \beta_i$ then every solution converges to a periodic solution of not necessarily prime period $2gcd(I_\beta \cup I_B)$. Hence all parts of the generalization are already verified with the exception of the unboundedness. It is therefore only necessary to generalize the results regarding the boundedness character of this equation. Fortunately the proof of unboundedness in [14] is sufficiently general so as not to require the condition $t = i - sr$. The authors only make use of the condition $t = i - sr$ during the proof of periodic convergence. Hence the generalized trichotomy result follows immediately by patching together the results in [2,10,12,13,14].

Theorem 1. *Consider the k^{th} order rational difference equation,*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, n \in \mathbb{N}. \tag{4}$$

Assume non-negative parameters and non-negative initial conditions so that the denominator is non-vanishing. Further assume that $\sum_{i=1}^{k} \beta_i > 0$ and $\alpha > 0$. Under these assumptions Equation (4) exhibits the following trichotomy behavior.

I. If $gcd(I_\beta \cup I_B) = 1$, i is even for all $i \in I_\beta$, and j is odd for all $j \in I_B$, then solutions of Equation (4) exhibit the following behavior.
   - *i. When $A > \sum_{i=1}^{k} \beta_i$ the unique equilibrium is globally asymptotically stable.*
   - *ii. When $A = \sum_{i=1}^{k} \beta_i$ every solution converges to a periodic solution of not necessarily prime period 2.*
   - *iii. When $0 \leq A < \sum_{i=1}^{k} \beta_i$ unbounded solutions exist for some initial conditions.*

II. If $gcd(I_\beta \cup I_B) \neq 1$ and after applying a change of variables i is even for all $i \in I_\beta$ and j is odd for all $j \in I_B$, then solutions of Equation (4) exhibit the following behavior.

i. When $A > \sum_{i=1}^k \beta_i$ the unique equilibrium is globally asymptotically stable.

ii. When $A = \sum_{i=1}^k \beta_i$ every solution converges to a periodic solution of not necessarily prime period $2gcd(I_\beta \cup I_B)$.

iii. When $0 \leq A < \sum_{i=1}^k \beta_i$ unbounded solutions exist for some initial conditions.

Proof. First notice that we have assumed $\sum_{i=1}^k \beta_i > 0$ and $\alpha > 0$. It is a natural consequence of conditions (I) and (II) that $\sum_{j=1}^k B_j > 0$. Assume for the sake of contradiction that $\sum_{j=1}^k B_j = 0$. Then it follows that $gcd(I_\beta \cup I_B) = gcd(I_\beta)$. After making a change of variables so that $gcd(I_\beta \cup I_B) = 1$, we necessarily have that $gcd(I_\beta) = 1$. This contradicts the statement that i is even for all $i \in I_\beta$. Since $\sum_{i=1}^k \beta_i > 0$, $\alpha > 0$, and $\sum_{j=1}^k B_j > 0$, the proof of the case (I.iii) is contained in [14] particularly Theorem 3.12. Notice that the case (II.iii) reduces to case (I.iii) after a change of variables. Particularly we use the change of variables contained in Remark 1 of [12].

The proof of the cases (I.i) and (II.i) is contained in [2,10], see also [12].

The proof of the case (I.ii) is contained in [2,13], see also [12]. It is clear that there exists a prime period 2 solution in this case. Simply choose all even initial conditions equal to q where $q > 0$ and $q \neq \sqrt{\frac{\alpha}{\sum_{j=1}^k B_j}}$. Then choose all odd initial conditions equal to $\frac{\alpha}{(\sum_{j=1}^k B_j)q}$.

The proof of the case (II.ii) follows from the prior case, case (I.ii), after a change of variables. The details regarding this change of variables are contained in Remark 1 of [12]. There is a computation required to prove that the period is $2gcd(I_\beta \cup I_B)$ after making said change of variables. Let us include this calculation.

We included in our assumption (II) that, after applying a change of variables, i is even for all $i \in I_\beta$ and j is odd for all $j \in I_B$. Hence we may use (I.ii) after applying our change of variables without further justification. From Remark 1 of [12] we see that we may write $x_{m\mu+L} = y_m^L$ for $\mu = gcd(I_\beta \cup I_B)$ and $L \in \{0, \ldots, \mu - 1\}$. Where $\{y_m^L\}$ is contained in case (I.ii). Hence $x_n - x_{n+2gcd(I_\beta \cup I_B)} = x_{m\mu+L} - x_{m\mu+L+2gcd(I_\beta \cup I_B)}$ where $\mu = gcd(I_\beta \cup I_B)$. So $x_{m\mu+L} - x_{m\mu+L+2gcd(I_\beta \cup I_B)} = x_{m\mu+L} - x_{(m+2)\mu+L}$ since $\mu = gcd(I_\beta \cup I_B)$. So $x_n - x_{n+2gcd(I_\beta \cup I_B)} = y_m^L - y_{m+2}^L$. Making use of (I.ii), $y_m^L - y_{m+2}^L$ converges to 0. Hence $x_n - x_{n+2gcd(I_\beta \cup I_B)}$ converges to 0. Hence in the case (II.ii) every solution converges to a periodic solution of not necessarily prime period $2gcd(I_\beta \cup I_B)$. □

4. Toward A New Trichotomy Result

As the title of this section indicates we work towards a new trichotomy result, particularly we prove the following theorem.

Theorem 2. *Consider the k^{th} order rational difference equation,*

$$x_n = \frac{\sum_{i=1}^k \beta_i x_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j}}, n \in \mathbb{N}. \tag{5}$$

Assume non-negative parameters and non-negative initial conditions. Further assume that $\sum_{i=1}^{k}\beta_i > 0$, $A > 0$, *and that there does not exist* $j \in I_B$ *so that* $gcd(I_\beta)|j$. *Under these assumptions solutions of Equation (5) exhibit the following trichotomy behavior.*

i. When $A > \sum_{i=1}^{k}\beta_i$ *the unique equilibrium is globally asymptotically stable.*
ii. When $A = \sum_{i=1}^{k}\beta_i$ *every solution converges to a periodic solution of not necessarily prime period* $gcd(I_\beta)$.
iii. When $0 < A < \sum_{i=1}^{k}\beta_i$ *unbounded solutions exist for some initial conditions.*

Proof. The cases (i) and (ii) were settled in [12]. Recall from [12] that we may choose initial conditions which produce a periodic solution of prime period $gcd(I_\beta)$.
We prove case (iii).
Choose initial conditions x_{-m} where $m \in \{0, \dots, k-1\}$ so that $x_{-m} = 1$ if $gcd(I_\beta)|m$ and $x_{-m} = 0$ otherwise.
Under this choice of initial conditions our solution $\{x_n\}$ has the property that $x_n > 0$ whenever $gcd(I_\beta)|n$ and $x_n = 0$ otherwise. We prove this using induction on n, our initial conditions provide the base case. Assume that the statement is true for all $n \leq N-1$. We show the statement for $n = N$.
This argument has two cases. First assume $gcd(I_\beta)|N$ then, the denominator is clearly non-zero since we assumed $A > 0$. Since we assumed $\sum_{i=1}^{k}\beta_i > 0$ we know that there exists $i \in I_\beta$ so that $\beta_i > 0$. It is sufficient to show $x_{N-i} > 0$. However since $i \in I_\beta$ it follows that $gcd(I_\beta)|i$. Thus $gcd(I_\beta)|N-i$. Hence by our induction hypothesis $x_{N-i} > 0$. We have shown the first case.
Now assume it is not true that $gcd(I_\beta)|N$. Again the denominator is clearly non-zero and furthermore,

$$x_N = \frac{\sum_{i=1}^{k}\beta_i x_{N-i}}{A + \sum_{j=1}^{k} B_j x_{N-j}} \leq \frac{\sum_{i=1}^{k}\beta_i x_{N-i}}{A}.$$

Take $i \in I_\beta$ it follows that $gcd(I_\beta)|i$. Hence by our assumption we have that for all $i \in I_\beta$ it is not true that $gcd(I_\beta)|N-i$. Indeed, assume $gcd(I_\beta)|N-i$ for some $i \in I_\beta$, then $gcd(I_\beta)|N$ contradicting our hypothesis. Hence by our induction hypothesis $x_{N-i} = 0$ for all $i \in I_\beta$. So $\sum_{i=1}^{k}\beta_i x_{N-i} = 0$. So $x_N = 0$ in this case.
Now we will make use of the prior result. Choose n such that $gcd(I_\beta)|n$. There does not exist $j \in I_B$ so that $gcd(I_\beta)|j$. Thus there does not exist $j \in I_B$ so that $gcd(I_\beta)|n-j$. Using this and the prior result, it follows that for n where $gcd(I_\beta)|n$, $x_{n-j} = 0$ for all $j \in I_B$. By our choice of n, we may write $m = \frac{n}{gcd(I_\beta)} \in \mathbb{N}$. So our equation reduces in this case to

$$x_{((m)gcd(I_\beta))} = \frac{\sum_{i=1}^{k}\beta_i x_{(m)gcd(I_\beta)-i}}{A} \geq \left(\frac{\sum_{i=1}^{k}\beta_i}{A}\right)\min_{i\in I_\beta}(x_{(m)gcd(I_\beta)-i})$$

$$\geq \left(\frac{\sum_{i=1}^{k}\beta_i}{A}\right)\min_{i\in\{1,\dots,\lfloor\frac{k}{gcd(I_\beta)}\rfloor\}}(x_{(m)gcd(I_\beta)-(igcd(I_\beta))}).$$

This is a difference inequality which holds for the subsequence $\{x_{(m)gcd(I_\beta)}\}$ for $m \geq k$. We now rename this subsequence and apply the methods used in [11]. We set $z_m =$

$x_{(m)gcd(I_\beta)}$ for $m \in \mathbb{N}$. As we have just shown $\{z_m\}$ satisfies the following difference inequality,

$$z_m \geq (\frac{\sum_{i=1}^k \beta_i}{A}) \min_{i \in \{1,\ldots,\lfloor \frac{k}{gcd(I_\beta)} \rfloor\}} (z_{m-i}), m \geq k.$$

Using the results of [11], particularly Theorem 3, we have that for $m \geq k$,

$$\min(z_{m-1}, \ldots, z_{m-\lfloor \frac{k}{gcd(I_\beta)} \rfloor}) \geq \min(y_{\lfloor \frac{m-k}{\lfloor \frac{k}{gcd(I_\beta)} \rfloor} \rfloor}, \ldots, y_{m-k}).$$

Where $\{y_m\}_{m=0}^\infty$ is a solution of the difference equation,

$$y_m = (\frac{\sum_{i=1}^k \beta_i}{A}) y_{m-1}, m \in \mathbb{N}. \tag{6}$$

With $y_0 = \min(z_{k-1}, \ldots, z_{k-\lfloor \frac{k}{gcd(I_\beta)} \rfloor})$.

Since we are in case (iii) we know that $0 < A < \sum_{i=1}^k \beta_i$ and so every positive solution diverges to ∞ for the simple difference equation (6). Hence using the inequality we have obtained, $\{z_m\}_{m=1}^\infty$ diverges to ∞. Hence with given initial conditions, there is a subsequence of our solution $\{x_n\}_{n=1}^\infty$, namely $\{x_{(mgcd(I_\beta))}\}_{m=1}^\infty$, which diverges to ∞. Hence our solution $\{x_n\}_{n=1}^\infty$ is unbounded. So we have exhibited an unbounded solution in the case (iii). This unbounded solution is independent of the choice of parameters in this case. Thus we have shown that when $0 < A < \sum_{i=1}^k \beta_i$ unbounded solutions exist for some initial conditions. □

Theorem 2 resolves a number of conjectures regarding boundedness character. Particularly the conjectures # 578,# 586,# 610,# 618 in [2].

5. Partial Results For The Remaining Case

In this section we provide many partial answers for Conjecture 1. During this section we use the ideas of modulo classes. Let us introduce these ideas in the following remark.

Remark 1. *We say that a is congruent to b with modulus c and write $a \equiv b \mod c$ if $c | a - b$. It is well known that given $z \in \mathbb{Z}$, there exists $a \in \{0, \ldots, c-1\}$ so that $z \equiv a \mod c$. We call such a the residue of z with respect to the modulus c, and write $a = z \mod c$.*

Notice that in the proof of unboundedness in case (iii) of Theorem 2 the problem was greatly simplified by choosing many of our initial conditions to be zero. We cannot do this here. If we choose certain initial conditions to be zero it is possible that we have chosen the initial conditions in such a way that eventually the denominator vanishes. To avoid this we make certain assumptions regarding the set of indices I_B. Let us describe these assumptions in full detail.

Condition 1. *We say that Condition 1 is satisfied if there exists $p \in \mathbb{N}$ so that $p | gcd(I_\beta)$ and there does not exist $j \in I_B$ so that $p | j$. We also require the existence of mutually disjoint sets $B, L, Z \subset \{0, \ldots, p-1\}$ with $B \neq \emptyset$ and with the following properties.*

(1) For all $b \in B$, $\{(b-j) \mod p : j \in I_B\} \subset L \cup Z$.
(2) For all $\ell \in L$, *there exists* $j \in I_B$ *so that* $(\ell - j) \mod p \in B$.
(3) For all $\eta \in \{0, \ldots, p-1\}$, $\{(\eta - j) \mod p : j \in I_B\} \cap Z \neq \{(\eta - j) \mod p : j \in I_B\}$.

Notice that this is a particularly intricate condition. It so happens that this is exactly the assumption required in order to make the subsequent argument. This being the case we find no need to simplify Condition 1. Instead we prove Theorem 3 by assuming Condition 1 and then we immediately prove Corollary 1. In Corollary 1 the assumptions are greatly simplified at the expense of generality.

Theorem 3. *Consider the* k^{th} *order rational difference equation,*

$$x_n = \frac{\sum_{i=1}^{k} \beta_i x_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j}}, n \in \mathbb{N}. \tag{7}$$

Assume non-negative parameters and non-negative initial conditions so that the denominator is non-vanishing. Further assume that $\sum_{i=1}^{k} \beta_i > 0$ *and that Condition 1 is satisfied for Equation (7). Then unbounded solutions of Equation (7) exist for some initial conditions.*

Proof. By assumption, we may choose $p \in \mathbb{N}$ and $B, L, Z \subset \{0, \ldots, p-1\}$ so that Condition 1 is satisfied. Choose initial conditions x_{-m} where $m \in \{0, \ldots, k-1\}$ so that the following holds. If $(-m \mod p) \in B$, then $x_{-m} > \frac{\sum_{i=1}^{k} \beta_i}{\min_{j \in I_B}(B_j)}$. If $(-m \mod p) \in Z$, then $x_{-m} = 0$. If $(-m \mod p) \in L$, then $x_{-m} < \frac{\sum_{i=1}^{k} \beta_i}{2\sum_{j=1}^{k} B_j}$. If $(-m \mod p) \notin Z$, then $x_{-m} > 0$.
Under this choice of initial conditions our solution $\{x_n\}$ has the following properties.
(a) $x_n > \frac{\sum_{i=1}^{k} \beta_i}{\min_{j \in I_B}(B_j)}$ whenever $(n \mod p) \in B$.
(b) $x_n = 0$ whenever $(n \mod p) \in Z$.
(c) $x_n < \frac{\sum_{i=1}^{k} \beta_i}{2\sum_{j=1}^{k} B_j}$ whenever $(n \mod p) \in L$.
(d) $x_n > 0$ whenever $(n \mod p) \notin Z$.
We prove this using induction on n, our initial conditions provide the base case. Assume that the statement is true for all $n \leq N-1$. We show the statement for $n = N$.
This induction proof has four cases. Let us begin by assuming $(N \mod p) \in B$.
Condition 1.1 tells us that in this case $\{(N-j) \mod p : j \in I_B\} \subset L \cup Z$. Hence $x_{N-j} < \frac{\sum_{i=1}^{k} \beta_i}{2\sum_{j=1}^{k} B_j}$ for all $j \in I_B$. Since $p | gcd(I_\beta)$, $N \mod p = (N-i) \mod p$ for all $i \in I_\beta$. So $x_{N-i} > \frac{\sum_{i=1}^{k} \beta_i}{\min_{j \in I_B}(B_j)}$ for all $i \in I_\beta$. Hence we have the following,

$$x_N = \frac{\sum_{i=1}^{k} \beta_i x_{N-i}}{\sum_{j=1}^{k} B_j x_{N-j}} \geq \frac{\sum_{i=1}^{k} \beta_i}{(\sum_{j=1}^{k} B_j)(\frac{\sum_{i=1}^{k} \beta_i}{2\sum_{j=1}^{k} B_j})}(\frac{\sum_{i=1}^{k} \beta_i}{\min_{j \in I_B}(B_j)}) \geq \frac{2\sum_{i=1}^{k} \beta_i}{\min_{j \in I_B}(B_j)}.$$

This inequality is obtained by simply replacing the terms in the denominator with their upper bound, and replacing the terms in the numerator with their lower bound. Also

note that x_N is well defined due to Condition 1.3. This finishes case (a).
We now assume $(N \mod p) \in Z$. Note that x_N is well defined due to Condition 1.3. Also note that since $p|gcd(I_\beta)$, $N \mod p = (N-i) \mod p$ for all $i \in I_\beta$. Hence $x_{N-i} = 0$ for all $i \in I_\beta$. So clearly,

$$x_N = \frac{\sum_{i=1}^k \beta_i x_{N-i}}{\sum_{j=1}^k B_j x_{N-j}} = 0.$$

This finishes case (b).
We now assume $(N \mod p) \in L$. Once again x_N is well defined due to Condition 1.3. Since $p|gcd(I_\beta)$, $N \mod p = (N-i) \mod p$ for all $i \in I_\beta$. Hence $x_{N-i} < \frac{\sum_{i=1}^k \beta_i}{2\sum_{j=1}^k B_j}$ for all $i \in I_\beta$. Condition 1.2 guarantees that there exists $j \in I_B$ so that $x_{N-j} > \frac{\sum_{i=1}^k \beta_i}{\min_{j\in I_B}(B_j)}$. Hence we have the following,

$$x_N = \frac{\sum_{i=1}^k \beta_i x_{N-i}}{\sum_{j=1}^k B_j x_{N-j}} < \frac{\sum_{i=1}^k \beta_i}{(\min_{j\in I_B}(B_j))(\frac{\sum_{i=1}^k \beta_i}{\min_{j\in I_B}(B_j)})}(\frac{\sum_{i=1}^k \beta_i}{2\sum_{j=1}^k B_j}) = \frac{\sum_{i=1}^k \beta_i}{2\sum_{j=1}^k B_j}.$$

This finishes case (c).
We now assume $(N \mod p) \notin Z$. Note that x_N is well defined due to Condition 1.3. Also note that since $p|gcd(I_\beta)$, $N \mod p = (N-i) \mod p$ for all $i \in I_\beta$. Hence $x_{N-i} > 0$ for all $i \in I_\beta$. So clearly,

$$x_N = \frac{\sum_{i=1}^k \beta_i x_{N-i}}{\sum_{j=1}^k B_j x_{N-j}} > 0.$$

This finishes case (d).
As we did in Theorem 2, we now use the facts we obtained from our induction to prove that a particular subsequence is unbounded. Take $b \in B$. We now show that $\{x_{mp+b}\}_{m=1}^\infty$ diverges to ∞. We explained earlier that $x_{mp+b-j} < \frac{\sum_{i=1}^k \beta_i}{2\sum_{j=1}^k B_j}$, since $\{(mp+b-j) \mod p : j \in I_B\} \subset L \cup Z$. Hence,

$$x_{mp+b} = \frac{\sum_{i=1}^k \beta_i x_{mp+b-i}}{\sum_{j=1}^k B_j x_{mp+b-j}} > \frac{\sum_{i=1}^k \beta_i x_{mp+b-i}}{(\sum_{j=1}^k B_j)(\frac{\sum_{i=1}^k \beta_i}{2\sum_{j=1}^k B_j})} \geq \frac{(\sum_{i=1}^k \beta_i)(\min_{i\in\{1,\dots,\lfloor \frac{k}{p}\rfloor\}}(x_{mp+b-ip}))}{(\sum_{j=1}^k B_j)(\frac{\sum_{i=1}^k \beta_i}{2\sum_{j=1}^k B_j})}$$
$$\geq 2 \min_{i\in\{1,\dots,\lfloor \frac{k}{p}\rfloor\}}(x_{mp+b-ip}), m \geq k.$$

This is a difference inequality which holds for the subsequence $\{x_{mp+b}\}$ for $m \geq k$. We now rename this subsequence and apply the methods used in [11]. We set $z_m = x_{mp+b}$ for $m \in \mathbb{N}$. As we have just shown $\{z_m\}$ satisfies the following difference inequality,

$$z_m \geq 2 \min_{i\in\{1,\dots,\lfloor \frac{k}{p}\rfloor\}}(z_{m-i}), m \geq k.$$

Using the results of [11], particularly Theorem 3, we have that for $m \geq k$,

$$\min(z_{m-1}, \dots, z_{m-\lfloor \frac{k}{p}\rfloor}) \geq \min(y_{\lfloor \frac{m-k}{\lfloor \frac{k}{p}\rfloor}\rfloor}, \dots, y_{m-k}).$$

Where $\{y_m\}_{m=0}^{\infty}$ is a solution of the difference equation,

$$y_m = 2y_{m-1}, m \in \mathbb{N}. \tag{8}$$

With $y_0 = \min(z_{k-1}, \dots, z_{k-\lfloor \frac{k}{p} \rfloor})$.
Clearly every positive solution diverges to ∞ for the simple difference equation (8). Hence using the inequality we have obtained, $\{z_m\}_{m=1}^{\infty}$ diverges to ∞. Hence with given initial conditions, there is a subsequence of our solution $\{x_n\}_{n=1}^{\infty}$, namely $\{x_{mp+b}\}_{m=1}^{\infty}$, which diverges to ∞. Hence our solution $\{x_n\}_{n=1}^{\infty}$ is unbounded. So we have exhibited an unbounded solution which is independent of the choice of parameters. □

Corollary 1. *Consider the k^{th} order rational difference equation,*

$$x_n = \frac{\sum_{i=1}^{k} \beta_i x_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j}}, n \in \mathbb{N}. \tag{9}$$

Assume non-negative parameters and non-negative initial conditions so that the denominator is non-vanishing. Further assume that $\sum_{i=1}^{k} \beta_i > 0$. Assume that there exists $p \in \mathbb{N}$ so that $p|gcd(I_\beta)$ and there does not exist $j \in I_B$ so that $p|j$. Also assume that there exist $j_1, j_2 \in I_B$, not necessarily distinct, so that $(j_1 + j_2) \mod p = 0$. Then unbounded solutions of Equation (9) exist for some initial conditions.

Proof. Using Theorem 3, it suffices to show that Condition 1 is satisfied under these assumptions. We know there exists $p \in \mathbb{N}$ so that $p|gcd(I_\beta)$ and there does not exist $j \in I_B$ so that $p|j$. Indeed, this is what we assumed to be true. Choose $B = \{0\}$, $L = \{-j_1 \mod p\}$, and $Z = \{-j \mod p : j \in I_B\} \sim \{-j_1 \mod p\}$.

(1) For all $b \in B$, namely $b = 0$, it is clear that $\{(0 - j) \mod p : j \in I_B\} \subset L \cup Z = (\{-j \mod p : j \in I_B\} \sim \{-j_1 \mod p\}) \cup \{-j_1 \mod p\} = \{(-j) \mod p : j \in I_B\}$.
(2) For all $\ell \in L$, namely $\ell = -j_1 \mod p$, there exists $j_2 \in I_B$ so that $(-j_1 - j_2) \mod p = (-(j_1 + j_2)) \mod p = 0 \mod p = 0 \in B$. This is since we assumed there exist $j_1, j_2 \in I_B$, not necessarily distinct, so that $(j_1 + j_2) \mod p = 0$.
(3) For all $\eta \in \{0, \dots, p-1\}$, $|\{(\eta - j) \mod p : j \in I_B\} \cap Z| \leq |Z| < |\{(\eta - j) \mod p : j \in I_B\}|$. So $\{(\eta - j) \mod p : j \in I_B\} \cap Z \neq \{(\eta - j) \mod p : j \in I_B\}$.

Thus Condition 1 is true under our prior assumptions. Hence the result follows. □

6. Cases of Small Order

Here we establish Conjecture 1 in its entirety for special cases of order $k \leq 3$. We also prove that Conjecture 1 holds for all special cases of order $k \leq 4$ excepting a single special case.

Theorem 4. *Consider the 4^{th} order rational difference equation,*

$$x_n = \frac{\sum_{i=1}^{4} \beta_i x_{n-i}}{\sum_{j=1}^{4} B_j x_{n-j}}, n \in \mathbb{N}. \tag{10}$$

Assume non-negative parameters and non-negative initial conditions so that the denominator is non-vanishing. Further assume that $\sum_{i=1}^{k}\beta_i > 0$ *and that there does not exist* $j \in I_B$ *so that* $gcd(I_\beta)|j$*. Also assume one of the following.*

(1) $I_\beta \neq \{3\}$.
(2) $I_B \neq \{1,4\}$.

Then unbounded solutions of Equation (10) exist for some initial conditions.

Proof. If $gcd(I_\beta) = 1$ then necessarily $gcd(I_\beta)|j$ for all $j \in I_B$ and hence the assumptions of this theorem are not satisfied. This theorem has three cases.

a. $gcd(I_\beta) = 2$.
b. $gcd(I_\beta) = 3$.
c. $gcd(I_\beta) = 4$.

In case (a), since $gcd(I_\beta) = 2$, there does not exist $j \in I_B$ so that $2|j$. Hence in this case $I_B \subset \{1,3\}$. Notice that $(1+3) \mod 2 = 4 \mod 2 = 0$, $(1+1) \mod 2 = 2 \mod 2 = 0$, and $(3+3) \mod 2 = 6 \mod 2 = 0$. So we may apply Corollary 1 in case (a).
In case (b), since $gcd(I_\beta) = 3$, there does not exist $j \in I_B$ so that $3|j$. Hence in this case $I_B \subset \{1,2,4\}$. If $I_B = \{1\}$, $I_B = \{2\}$, or $I_B = \{4\}$ then the equation is linear after a change of variables. Hence, in these cases, Equation (10) has unbounded solutions for some initial conditions. If $I_B = \{1,2\}$, $I_B = \{2,4\}$, or $I_B = \{1,2,4\}$ then Corollary 1 applies. We assumed that $I_B \neq \{1,4\}$ when $gcd(I_\beta) = 3$. This is the unsolved case.
In case (c), since $gcd(I_\beta) = 4$, there does not exist $j \in I_B$ so that $4|j$. Hence in this case $I_B \subset \{1,2,3\}$. If $2 \in I_B$ then Corollary 1 applies. Hence in the remaining cases $I_B \subset \{1,3\}$. If $I_B = \{1\}$ or $I_B = \{3\}$ then the equation is linear after a change of variables. Hence, in these cases, Equation (10) has unbounded solutions for some initial conditions. If $I_B = \{1,3\}$, notice that $(1+3) \mod 4 = 4 \mod 4 = 0$. So we may apply Corollary 1 in this case. □

There remains a single case of order 4 for which Conjecture 1 is still not verified. We present this case in the following conjecture.

Conjecture 2. *Consider the* 4^{th} *order rational difference equation,*

$$x_n = \frac{x_{n-3}}{Bx_{n-1} + x_{n-4}}, n \in \mathbb{N}. \tag{11}$$

We conjecture that unbounded solutions of Equation (11) exist for some initial conditions.

We feel that Conjecture 2 is a good starting point for further work. This is the last remaining case where Conjecture 1 has not been verified for rational difference equations of order 4.
Notice that we have confirmed the conjectures #296, #608, and #616 in [2] regarding the boundedness character of rational difference equations.

7. Consequences

Here we focus on the following question posed in [2]; Do the dominant roots of the characteristic equation of the linearized equation of Equation (1) about its equilibrium provide an alternative method for predicting this trichotomy behavior? The authors in

[2] ask this question regarding all periodic trichotomies of order 3. We now examine this question in retrospect for the trichotomy results discussed prior. We observe the following phenomenon.

Theorem 5. *Consider the k^{th} order rational difference equation,*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, n \in \mathbb{N}. \tag{12}$$

Assume non-negative parameters and non-negative initial conditions so that the denominator is non-vanishing. Further assume that $\sum_{i=1}^{k} \beta_i > 0$. For ease of notation let $p = gcd(I_\beta \cup I_B)$ and $q = gcd(I_\beta)$. Under these assumptions the dominant roots of the characteristic equation of the linearized equation of Equation (12) about its equilibrium are as follows.

I. If $\alpha > 0$, $gcd(I_\beta \cup I_B) = 1$, i is even for all $i \in I_\beta$, and j is odd for all $j \in I_B$, then we have the folowing.
 - *i. When $A > \sum_{i=1}^{k} \beta_i$ all roots of the characteristic equation lie within $\mathbb{D}$.*
 - *ii. When $A = \sum_{i=1}^{k} \beta_i$ there is a dominant root of the characteristic equation which is a square root of unity.*

II. If $\alpha > 0$, $gcd(I_\beta \cup I_B) \neq 1$ and after applying a change of variables i is even for all $i \in I_\beta$ and j is odd for all $j \in I_B$, then we have the folowing.
 - *i. When $A > \sum_{i=1}^{k} \beta_i$ all roots of the characteristic equation lie within $\mathbb{D}$.*
 - *ii. When $A = \sum_{i=1}^{k} \beta_i$ there are p dominant roots of the characteristic equation which are $2p^{th}$ roots of unity.*

III. If $\alpha = 0$ and there does not exist $j \in I_B$ so that $gcd(I_\beta)|j$, then we have the folowing.
 - *i. When $A > \sum_{i=1}^{k} \beta_i$ all roots of the characteristic equation lie within $\mathbb{D}$.*
 - *ii. When $A = \sum_{i=1}^{k} \beta_i$ there are q dominant roots of the characteristic equation which are q^{th} roots of unity.*

Proof. First notice that $I_\beta \cap I_B = \emptyset$ for all of the cases above. This significantly reduces the difficulty regarding the computation of the the characteristic equation of the linearized equation of Equation (12) about its equilibrium in each case. For details on linearized stability analysis see Section 1.2 of [2]. A relatively simple computation, combined with the fact that $I_\beta \cap I_B = \emptyset$, reveals that in the cases (III.i) and (III.ii) the characteristic equation is as follows.

$$\lambda^k - \sum_{i \in I_\beta} \frac{\beta_i \lambda^{k-i}}{A} = 0. \tag{13}$$

To prove case (III.i) we use Rouché's theorem. Notice that for ζ such that $|\zeta| = 1$ and $A > \sum_{i=1}^{k} \beta_i$ the following holds.

$$|(\zeta^k - \sum_{i \in I_\beta} \frac{\beta_i \zeta^{k-i}}{A}) - \zeta^k| = |\sum_{i \in I_\beta} \frac{\beta_i \zeta^{k-i}}{A}| \leq \sum_{i \in I_\beta} \frac{\beta_i |\zeta|^{k-i}}{A} = \sum_{i \in I_\beta} \frac{\beta_i}{A} < 1 = |\zeta|^k = |\zeta^k|.$$

Hence, by Rouché's theorem, when $A > \sum_{i=1}^{k} \beta_i$ all roots lie within $\mathbb{D}$.
To prove case (III.ii) we first notice that for any q^{th} root of unity, λ_q, $\lambda_q^n = \lambda_q^{n+q}$. Hence for all $n \in \mathbb{N}$, $\lambda_q^n = \lambda_q^{n \mod q}$. Since $q = gcd(I_\beta)$, $k \mod q = (k-i) \mod q$ for all $i \in I_\beta$. Hence when $A = \sum_{i=1}^{k} \beta_i$ and λ_q is a q^{th} root of unity in Equation (13) we have the following.

$$\lambda_q^k - \sum_{i \in I_\beta} \frac{\beta_i \lambda_q^{k-i}}{A} = (\lambda_q^{k \mod q})(1 - \sum_{i \in I_\beta} \frac{\beta_i}{A}) = (\lambda_q^{k \mod q})(0) = 0.$$

Hence all q^{th} roots of unity are roots of Equation (13) when $A = \sum_{i=1}^{k} \beta_i$.
Using Rouché's theorem notice that for ζ such that $|\zeta| = 1 + \epsilon$ where $\epsilon > 0$, and $A = \sum_{i=1}^{k} \beta_i$ the following holds.

$$|(\zeta^k - \sum_{i \in I_\beta} \frac{\beta_i \zeta^{k-i}}{A}) - \zeta^k| = |\sum_{i \in I_\beta} \frac{\beta_i \zeta^{k-i}}{A}| \leq \sum_{i \in I_\beta} \frac{\beta_i |\zeta|^{k-i}}{A}$$

$$\leq (\sum_{i \in I_\beta} \frac{\beta_i}{A})(|\zeta|^{k-1}) = (1+\epsilon)^{k-1} < (1+\epsilon)^k = |\zeta|^k = |\zeta^k|.$$

Hence by Rouché's theorem all roots of Equation (13) are within $\bar{\mathbb{D}}$. Thus the q^{th} roots of unity are dominant roots of the characteristic equation.
We now consider cases (I) and (II). For ease of notation we let $\mathbf{C} = \sum_{i=1}^{k} \beta_i$ and $\mathbf{D} = \sum_{j=1}^{k} B_j$. A relatively simple computation, combined with the fact that $I_\beta \cap I_B = \emptyset$, reveals that in the cases (I) and (II) the characteristic equation is as follows.

$$\lambda^k - \sum_{i \in I_\beta} \frac{\beta_i \lambda^{k-i}}{A + \mathbf{D}\bar{x}} + \sum_{j \in I_B} \frac{(\alpha + \mathbf{C}\bar{x}) B_j \lambda^{k-j}}{(A + \mathbf{D}\bar{x})^2} = 0. \tag{14}$$

Where

$$\bar{x} = \frac{(\mathbf{C} - A) + \sqrt{(\mathbf{C} - A)^2 + 4\alpha\mathbf{D}}}{2\mathbf{D}}.$$

We first prove the following inequality which holds when $A > \sum_{i=1}^{k} \beta_i$. This will simplify things later.

$$\alpha\mathbf{D} + (\mathbf{C} - A)(A + \mathbf{D}\bar{x}) < (\mathbf{D}\bar{x})^2. \tag{15}$$

This may be rewritten,

$$\alpha\mathbf{D} + (\mathbf{C} - A)(A + \mathbf{D}\bar{x}) < (\frac{(\mathbf{C} - A) + \sqrt{(\mathbf{C} - A)^2 + 4\alpha\mathbf{D}}}{2})^2.$$

Which is true if and only if,

$$4\alpha\mathbf{D} + 4(\mathbf{C} - A)(A + \mathbf{D}\bar{x}) < 2(\mathbf{C} - A)^2 + 4\alpha\mathbf{D} + 2(\mathbf{C} - A)\sqrt{(\mathbf{C} - A)^2 + 4\alpha\mathbf{D}}).$$

The prior statement is true if and only if,

$$2(\mathbf{C} - A)(A + \mathbf{D}\bar{x}) < (\mathbf{C} - A)^2 + (\mathbf{C} - A)\sqrt{(\mathbf{C} - A)^2 + 4\alpha\mathbf{D}}).$$

Since $A > \sum_{i=1}^k \beta_i$ the prior statement is true if and only if,

$$2(A+\mathbf{D}\bar{x}) > (\mathbf{C}-A) + \sqrt{(\mathbf{C}-A)^2 + 4\alpha\mathbf{D}}) = 2\mathbf{D}\bar{x}.$$

However the prior statement must hold since $A > \sum_{i=1}^k \beta_i > 0$. Thus we have shown that Equation (15) holds whenever $A > \sum_{i=1}^k \beta_i$.
We now use Rouché's theorem and Equation (15) to show (I.i) and (II.i).Notice that for ζ such that $|\zeta| = 1$ and $A > \sum_{i=1}^k \beta_i$ the following holds.

$$|(\zeta^k - \sum_{i\in I_\beta} \frac{\beta_i \zeta^{k-i}}{A+\mathbf{D}\bar{x}} + \sum_{j\in I_B} \frac{(\alpha+\mathbf{C}\bar{x})B_j\zeta^{k-j}}{(A+\mathbf{D}\bar{x})^2}) - \zeta^k| = |-\sum_{i\in I_\beta} \frac{\beta_i \zeta^{k-i}}{A+\mathbf{D}\bar{x}} + \sum_{j\in I_B} \frac{(\alpha+\mathbf{C}\bar{x})B_j\zeta^{k-j}}{(A+\mathbf{D}\bar{x})^2}|$$

$$\leq \sum_{i\in I_\beta} \frac{\beta_i |\zeta|^{k-i}}{A+\mathbf{D}\bar{x}} + \sum_{j\in I_B} \frac{(\alpha+\mathbf{C}\bar{x})B_j|\zeta|^{k-j}}{(A+\mathbf{D}\bar{x})^2} = \sum_{i\in I_\beta} \frac{\beta_i}{A+\mathbf{D}\bar{x}} + \sum_{j\in I_B} \frac{(\alpha+\mathbf{C}\bar{x})B_j}{(A+\mathbf{D}\bar{x})^2}$$

$$= \frac{\mathbf{C}}{A+\mathbf{D}\bar{x}} + \frac{(\alpha+\mathbf{C}\bar{x})\mathbf{D}}{(A+\mathbf{D}\bar{x})^2} = \frac{\mathbf{C}(A+2\mathbf{D}\bar{x}) + \alpha\mathbf{D}}{(A+\mathbf{D}\bar{x})^2} < \frac{A^2 + 2A\mathbf{D}\bar{x} + (\alpha\mathbf{D} + (\mathbf{C}-A)(A+\mathbf{D}\bar{x}))}{(A+\mathbf{D}\bar{x})^2}.$$

Now we use Equation (15) and finish the proof.

$$\frac{A^2 + 2A\mathbf{D}\bar{x} + (\alpha\mathbf{D} + (\mathbf{C}-A)(A+\mathbf{D}\bar{x}))}{(A+\mathbf{D}\bar{x})^2} < \frac{A^2 + 2A\mathbf{D}\bar{x} + (\mathbf{D}\bar{x})^2}{(A+\mathbf{D}\bar{x})^2} = 1 = |\zeta|^k = |\zeta^k|.$$

Hence, by Rouché's theorem, when $A > \sum_{i=1}^k \beta_i$ all roots lie within $\mathbb{D}$. This proves the cases (I.i) and (II.i).
All that remain are the cases (I.ii) and (II.ii) where $A = \sum_{i=1}^k \beta_i$. Hence $\bar{x}$ simplifies in the following way.

$$\bar{x} = \sqrt{\frac{\alpha}{\mathbf{D}}}.$$

Thus Equation (14) simplifies to become the following equation.

$$\lambda^k - \sum_{i\in I_\beta} \frac{\beta_i \lambda^{k-i}}{A+\sqrt{\alpha\mathbf{D}}} + \sum_{j\in I_B} \frac{(\alpha + A\sqrt{\frac{\alpha}{\mathbf{D}}})B_j\lambda^{k-j}}{(A+\sqrt{\alpha\mathbf{D}})^2} = 0. \tag{16}$$

We first show that in case (I.i) -1 is a root of Equation (16). Since we assumed i is even for all $i \in I_\beta$ and j is odd for all $j \in I_B$, substituting $\lambda = -1$ into Equation (16) yields the following.

$$(-1)^k - \sum_{i\in I_\beta} \frac{\beta_i (-1)^{k-i}}{A+\sqrt{\alpha\mathbf{D}}} + \sum_{j\in I_B} \frac{(\alpha + A\sqrt{\frac{\alpha}{\mathbf{D}}})B_j(-1)^{k-j}}{(A+\sqrt{\alpha\mathbf{D}})^2}$$

$$= (-1)^k(1 - \frac{\mathbf{C}}{A+\sqrt{\alpha\mathbf{D}}} - \frac{(\alpha + A\sqrt{\frac{\alpha}{\mathbf{D}}})\mathbf{D}}{(A+\sqrt{\alpha\mathbf{D}})^2})$$

$$= (-1)^k(\frac{(A+\sqrt{\alpha\mathbf{D}})^2 - \mathbf{C}A - \mathbf{C}\sqrt{\alpha\mathbf{D}} - \alpha\mathbf{D} - A\sqrt{\alpha\mathbf{D}}}{(A+\sqrt{\alpha\mathbf{D}})^2})$$

$$= (-1)^k(\frac{(A+\sqrt{\alpha\mathbf{D}})^2 - A^2 - 2A\sqrt{\alpha\mathbf{D}} - \alpha\mathbf{D}}{(A+\sqrt{\alpha\mathbf{D}})^2}) = (-1)^k(0) = 0.$$

So -1 is a root of Equation (16) in case (I.ii). We have just shown that in case (I.ii) there is a root of Equation (16) which is a square root of unity. Later we show that it is a dominant root.

We now turn our attention to the case (II.ii). Consider roots λ_p so that $\lambda_p^p = -1$. Such roots are $2p^{th}$ roots of unity. Hence $\lambda_p^n = \lambda_p^{n+2p}$. Hence for all $n \in \mathbb{N}$, $\lambda_p^n = \lambda_p^{n \mod 2p}$. Recall that $p = gcd(I_\beta \cup I_B)$. Also recall that, by assumption, after applying a change of variables i is even for all $i \in I_\beta$ and j is odd for all $j \in I_B$. This means that $i \mod 2p = 0$ for all $i \in I_\beta$ and $j \mod 2p = p$ for all $j \in I_B$. Hence $k \mod 2p = (k-i) \mod 2p$ for all $i \in I_\beta$ and $k \mod 2p = (((k-j) \mod 2p) + p) \mod 2p$ for all $j \in I_B$. Thus $\lambda_p^k = \lambda_p^{k-i}$ for all $i \in I_\beta$ and $\lambda_p^k = -\lambda_p^{k-j}$ for all $j \in I_B$. Hence substituting λ_p into Equation (16) in the case (II.ii) we have the following.

$$\lambda_p^k - \sum_{i\in I_\beta} \frac{\beta_i \lambda_p^{k-i}}{A+\sqrt{\alpha \mathbf{D}}} + \sum_{j \in I_B} \frac{(\alpha + A\sqrt{\frac{\alpha}{\mathbf{D}}})B_j \lambda_p^{k-j}}{(A+\sqrt{\alpha\mathbf{D}})^2}$$

$$= (\lambda_p^k)(1 - \frac{\mathbf{C}}{A+\sqrt{\alpha\mathbf{D}}} - \frac{(\alpha + A\sqrt{\frac{\alpha}{\mathbf{D}}})\mathbf{D}}{(A+\sqrt{\alpha\mathbf{D}})^2})$$

$$= (\lambda_p^k)(\frac{(A+\sqrt{\alpha\mathbf{D}})^2 - A^2 - 2A\sqrt{\alpha\mathbf{D}} - \alpha\mathbf{D}}{(A+\sqrt{\alpha\mathbf{D}})^2}) = (\lambda_p^k)(0) = 0.$$

Hence all λ_p such that $\lambda_p^p = -1$ are roots of Equation (16) in case (II.ii). We have just shown that in case (II.ii) there are p roots of Equation (16) which are $2p^{th}$ roots of unity. We must now show that in the cases (I.ii) and (II.ii) all roots of Equation (16) lie in $\bar{\mathbb{D}}$. This will guarantee that the roots found earlier are dominant roots. To do this we use Rouché's theorem. Notice that for ζ such that $|\zeta| = 1+\epsilon$ where $\epsilon > 0$, and $A = \sum_{i=1}^k \beta_i$ the following holds.

$$|(\zeta^k - \sum_{i\in I_\beta} \frac{\beta_i \zeta^{k-i}}{A+\sqrt{\alpha\mathbf{D}}} + \sum_{j\in I_B} \frac{(\alpha + A\sqrt{\frac{\alpha}{\mathbf{D}}})B_j\zeta^{k-j}}{(A+\sqrt{\alpha\mathbf{D}})^2}) - \zeta^k|$$

$$= |-\sum_{i\in I_\beta} \frac{\beta_i \zeta^{k-i}}{A+\sqrt{\alpha\mathbf{D}}} + \sum_{j\in I_B} \frac{(\alpha + A\sqrt{\frac{\alpha}{\mathbf{D}}})B_j\zeta^{k-j}}{(A+\sqrt{\alpha\mathbf{D}})^2}|$$

$$\leq \sum_{i\in I_\beta} \frac{\beta_i |\zeta|^{k-i}}{A+\sqrt{\alpha\mathbf{D}}} + \sum_{j\in I_B} \frac{(\alpha + A\sqrt{\frac{\alpha}{\mathbf{D}}})B_j|\zeta|^{k-j}}{(A+\sqrt{\alpha\mathbf{D}})^2}$$

$$\leq (1+\epsilon)^{k-1}(\frac{\mathbf{C}}{A+\sqrt{\alpha\mathbf{D}}} + \frac{\alpha\mathbf{D} + A\sqrt{\alpha\mathbf{D}}}{(A+\sqrt{\alpha\mathbf{D}})^2}) = (1+\epsilon)^{k-1}(\frac{A^2 + \alpha\mathbf{D} + 2A\sqrt{\alpha\mathbf{D}}}{(A+\sqrt{\alpha\mathbf{D}})^2})$$

$$= (1+\epsilon)^{k-1} < (1+\epsilon)^k = |\zeta|^k = |\zeta^k|.$$

Hence by Rouché's theorem all roots of Equation (16) are within $\bar{\mathbb{D}}$. Thus in the case (I.ii) the root -1 is a dominant root of the characteristic equation which is a square root of unity. Furthermore in the case (II.ii) the roots λ_p are p dominant roots of the characteristic equation which are $2p^{th}$ roots of unity.

□

In [2] Camouzis and Ladas ask about the dominant roots of the characteristic equation of the linearized equation of Equation (1) about its equilibrium. Particularly they ask whether trichotomy behavior can be predicted from these roots. Hindsight grants us insight into a particular phenomenon. Notice that in Theorem 5, in the case where $A = \sum_{i=1}^{k} \beta_i$, the following takes place. When (I) holds there are square roots of unity which are dominant roots, but also we have shown earlier that every solution converges to a periodic solution of period 2. When (II) holds there are $2gcd(I_\beta \cup I_B)^{th}$ roots of unity which are dominant roots, but also we have shown earlier that every solution converges to a periodic solution of period $2gcd(I_\beta \cup I_B)$. When (III) holds there are $gcd(I_\beta)^{th}$ roots of unity which are dominant roots, but also we have shown earlier that every solution converges to a periodic solution of period $gcd(I_\beta)$. Looking retrospectively lends credence toward the possibility of predicting trichotomy behavior from the roots of the characteristic equation. We leave such investigations of this relationship for further work. It is worthwhile to note that if Conjecture 1 holds then Conjecture 5.225.3 in [2] is satisfied whenever $\sum_{i=1}^{k} \beta_i > 0$ and one of the conditions (I), (II), or (III) holds.

8. Conclusion

In the previous sections we have demonstrated several trichotomy results, but we have also left many unanswered questions. Conjecture 2 provides a good starting point for further work. The solution of Conjecture 2 may provide new insights leading to the resolution of Conjecture 1. The ideas regarding the dominant roots of the characteristic equation of the linearized equation of Equation (1) about its equilibrium were primarily motivated by the remarks in [2]. These ideas may also lead in interesting directions. Before concluding we note that the results in [2] are a compilation of the combined work of many authors. We encourage the reader to refer to the citation list in [2] for a full account. We list, for the convienience of the reader, several references used in [2] which are particularly relevant to periodic trichotomies, namely [1],[6],[9], and [7].

References

[1] A.M. Amleh, D.A. Georgiou, E.A. Grove, and G. Ladas, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$, *J. Math. Anal. Appl.* **233**(1999), 790-798.

[2] E. Camouzis and G. Ladas, *Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC Press, Boca Raton, 2007.

[3] E. Chatterjee, E.A. Grove, Y. Kostrov, and G. Ladas, On the Trichotomy character of $x_{n+1} = \frac{\alpha+\gamma x_{n-1}}{A+Bx_n+x_{n-2}}$, *J. Difference Equa. Appl.* **9**(2003), 1113-1128.

[4] H.A. El-Metwally, E.A. Grove, and G. Ladas, A global convergence result with applications to periodic solutions, *J. Math. Anal. Appl.* **245**(2000), 161-170.

[5] H.A. El-Metwally, E.A. Grove, G. Ladas, and H.D. Voulov, On the global attractivity and the periodic character of some difference equations, *J. Difference Equa. Appl.* **7**(2001), 837-850.

[6] C.H. Gibbons, M.R.S. Kulenović, and G. Ladas, On the recursive sequence $x_{n+1} = \frac{\alpha+\beta x_{n-1}}{\gamma+x_n}$, *Math. Sci. Res. Hot-Line* **4**(2000), 1-11.

[7] E.A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman & Hall/CRC Press, Boca Raton, 2005.

[8] E.A. Grove, G. Ladas, M. Predescu, and M. Radin, On the global character of the difference equation $x_{n+1} = \frac{\alpha+\gamma x_{n-(2k+1)}+\delta x_{n-2l}}{A+x_{n-2l}}$, *J. Difference Equa. Appl.* **9**(2003), 171-200.

[9] G.L. Karakostas and S. Stevic, On the recursive sequence $x_{n+1} = B + \frac{x_{n-k}}{a_0 x_n+\cdots+a_{k-1}x_{n-k+1}+\gamma}$, *J. Difference Equa. Appl.* **10**(2004), 809-815.

[10] V.L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[11] F.J. Palladino, Difference inequalities, comparison tests, and some consequences, *Involve J. Math.* **1**(2008), 91-100.

[12] F.J. Palladino, On the Characterization of Rational Difference Equations, *J. Difference Equa. Appl.* **14**(2008), (to appear).

[13] S. Stević, On the recursive sequence $x_{n+1} = \frac{\alpha+\sum_{i=1}^{k}\alpha_i x_{n-p_i}}{1+\sum_{j=1}^{m}\beta_j x_{n-q_j}}$, *J. Difference Equa. Appl.* **13**(2007), 41-46.

[14] Q. Wang, F. Zeng, G. Zang, and X. Liu, Dynamics of the difference equation $x_{n+1} = \frac{\alpha+B_1x_{n-1}+B_3x_{n-3}+\cdots+B_{2k+1}x_{n-2k-1}}{A+B_0x_n+B_2x_{n-2}+\cdots+B_{2k}x_{n-2k}}$, *J. Difference Equa. Appl.* **12**(2006), 399-417.