ON THE CHARACTERIZATION OF RATIONAL DIFFERENCE EQUATIONS

FRANK J. PALLADINO

Abstract. We explore the implications of monotonic character for difference equations of order greater than one. Several techniques are developed culminating in the complete characterization of the behavior of solutions to the k^{th} order rational difference equation

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N},$$

in the case $A \geq \sum_{i=1}^{k} \beta_i$. As is customary we assume non-negative parameters and non-negative initial conditions.

fpalladino@math.sunysb.edu

1. Introduction

In [1], Camouzis and Ladas present a compendium of knowledge regarding the k^{th} order rational difference equation with non-negative parameters and non-negative initial conditions

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, n \in \mathbb{N}. \tag{1}$$

The authors develop several innovative techniques which serve to unify and expand convergence results for rational difference equations. Particularly we are referring to theorems 1.6.7-1.6.11 in [1]. These results make extensive use of the monotonic character inherent in the k^{th} order rational difference equation. In Section 2, inspired by these results we move into a more general framework which allows the monotonic properties of our difference equation to naturally exhibit themselves.

In Section 3, armed with these new techniques we are able to ascertain the behaviour of solutions in several interesting cases which have thus far eluded proof. In so doing we justify our prior abstraction and complete the characterization of solutions to the k^{th} order rational difference equation in the case $A \geq \sum_{i=1}^{k} \beta_i$.

Date: April 5, 2008.

[0] **Keywords: difference equation, periodic convergence, global stability, monotonic character.**

AMS Subject Classification: 39A10,39A11

2. Monotonicity Results

It behooves us to accommodate more general monotonic behavior utilizing similar principles as those applied in [1]. We ascertain this knowledge through the use of a function, h, chosen at the discretion of the reader so as to be in concordance with our recursive function f. The function, h, allows us to narrow our gaze to a particular set Ω as described in the following theorem. This is beneficial as the set Ω is often easier to understand than an arbitrary solution to our recursive equation.

Theorem 1. *Consider a metric space X. Let* $h : X \rightarrow [0,\infty)$ *be continuous. Let* $f : X \rightarrow X$ *be continuous. Let* $h \circ f(x) \leq h(x)$ *for each* $x \in X$*. Let* $\Omega = \{x \in X | h \circ f^j(x) = h(x)$ *for all* $j \in \mathbb{N}\}$*. Let* $\{x_n\}_{n=0}^{\infty}$ *be a solution to the recursive equation*

$$x_n = f(x_{n-1}), n \in \mathbb{N}, x_0 \in X. \tag{2}$$

Let $\{x_{n_k}\}_{k=1}^{\infty}$ *be a convergent subsequence of* $\{x_n\}_{n=0}^{\infty}$*, and let* $lim_{k\rightarrow\infty} x_{n_k} = a$*.*
Then $a \in \Omega$*.*

Proof. Since $h \circ f(x) \leq h(x)$, for $\{x_n\}_{n=0}^{\infty}$ we have $h(x_{n+1}) \leq h(x_n)$ for all $n \in \mathbb{N}$. So $\{h(x_n)\}_{n=0}^{\infty}$ is a bounded monotonic sequence of real numbers and thus converges. Hence for all $j \in \mathbb{N}$
$h \circ f^j(a) = h \circ f^j(lim_{k\rightarrow\infty} x_{n_k}) = lim_{k\rightarrow\infty} h(x_{n_k+j}) = lim_{k\rightarrow\infty} h(x_{n_k}) = h(a)$
So $a \in \Omega$. □

The set Ω is of paramount importance. Not only does Ω contain all subsequential limits of solutions to our recursive equation, but also the properties of Ω have a deep relationship with periodic convergence of solutions to our recursive equation.

Theorem 2. *Consider a metric space X. Let* $h : X \rightarrow [0,\infty)$ *be continuous. Let* $f : X \rightarrow X$ *be continuous. Let* $h \circ f(x) \leq h(x)$ *for each* $x \in X$*. Let* $\Omega = \{x \in X | h \circ f^j(x) = h(x)$ *for all* $j \in \mathbb{N}\}$*. Let* $\{x_n\}_{n=0}^{\infty}$ *be a solution to the recursive equation,*

$$x_n = f(x_{n-1}), n \in \mathbb{N}, x_0 \in X. \tag{3}$$

Let every subsequence of $\{x_n\}_{n=0}^{\infty}$ *have a convergent subsequence.*
Let $h^{-1}(\{lim_{n\rightarrow\infty} h(x_n)\}) \cap \Omega$ *be finite.*
Then $\{x_n\}_{n=0}^{\infty}$ *converges to a periodic solution in* $h^{-1}(\{lim_{n\rightarrow\infty} h(x_n)\}) \cap \Omega$*.*

Proof. For the sake of notation, let us denote $h^{-1}(\{lim_{n\rightarrow\infty} h(x_n)\}) \cap \Omega = \Omega_L$. We claim that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that $min_{a\in\Omega_L}(d(x_n, a)) < \epsilon$ for $n \geq N$. Suppose not, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ so that $min_{a\in\Omega_L}(d(x_{n_k}, a)) \geq \epsilon$ for all $k \in \mathbb{N}$. However there is a further convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$, by our assumptions, so that $lim_{j\rightarrow\infty} x_{n_j} = b \notin \Omega_L$, but from Theorem 1, $b \in \Omega$. Hence $b \notin h^{-1}(\{lim_{n\rightarrow\infty} h(x_n)\})$, so $lim_{j\rightarrow\infty} h(x_{n_j}) = h(b) \neq lim_{n\rightarrow\infty} h(x_n)$. This is a contradiction.
Our function, f, is continuous and Ω_L is finite. It quickly follows from the definition of continuity that, given $\epsilon > 0$, there exists $\delta > 0$ so that for all $a \in \Omega_L$ and $x \in X$, whenever $d(x, a) < \delta$, $d(f(x), f(a)) < \epsilon$.
Let

$$\rho = \min_{a_1,a_2\in\Omega_L, a_1\neq a_2} d(a_1,a_2)/3 > 0.$$

Given $\rho > 0$ there exists $\delta_\rho > 0$ so that for all $a \in \Omega_L$ and $x \in X$, whenever $d(x,a) < \delta_\rho$, $d(f(x), f(a)) < \rho$. Take $\delta = min(\rho, \delta_\rho) > 0$.
The set Ω_L is, by definition, invariant under f. Since Ω_L is finite this implies that any $a \in \Omega_L$ is eventually periodic with period less than or equal to $|\Omega_L|$.
Using our earlier claim, there exists $N \in \mathbb{N}$ so that $min_{a\in\Omega_L}(d(x_n,a)) < \delta$ for $n \geq N$. Particularly, $min_{a\in\Omega_L}(d(x_N,a)) < \delta$. So $d(x_N,a_1) < \delta$ for some $a_1 \in \Omega_L$.
This provides the base case for an induction on $J \in \mathbb{N}$. Assume that $d(x_{N+J-1}, f^{J-1}(a_1)) < \delta$. Since $d(x_{N+J-1}, f^{J-1}(a_1)) < \delta \leq \delta_\rho$, $d(x_{N+J}, f^J(a_1)) < \rho$. For $b \in \Omega_L$, where $b \neq f^J(a_1)$, by the triangle inequality and the definition of ρ we have $3\rho \leq d(b, f^J(a_1)) \leq d(x_{N+J}, f^J(a_1)) + d(x_{N+J}, b) < \rho + d(x_{N+J}, b)$. Hence $d(x_{N+J}, b) > 2\rho > \delta$ for all $b \in \Omega_L$ where $b \neq f^J(a_1)$. However $min_{a\in\Omega_L}(d(x_{N+J},a)) < \delta$, thus $d(x_{N+J}, f^J(a_1)) < \delta$.
Hence for all $J \in \mathbb{N}$, $d(x_{N+J}, b) > \delta$ for all $b \in \Omega_L$ where $b \neq f^J(a_1)$. From our claim, given $\epsilon > 0$ there exists $M_1 \in \mathbb{N}$ so that $min_{a\in\Omega_L}(d(x_n,a)) < \epsilon$ for $n \geq M_1$. Thus given $\epsilon > 0$ there exists $M_2 \in \mathbb{N}$ so that $d(x_{N+J}, f^J(a_1)) < \epsilon$ for $J \geq M_2$. Recall $\{f^n(a_1)\}_{n=0}^\infty$ is an eventually periodic solution in Ω_L. Hence $\{x_n\}_{n=0}^\infty$ converges to a periodic solution in $h^{-1}(\{lim_{n\to\infty} h(x_n)\}) \cap \Omega$. □

3. Applications to Rational Difference Equations

Our goal is to completely characterize the behavior of solutions of the k^{th} order rational difference equation, with non-negative parameters and non-negative initial conditions,

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j}}, n \in \mathbb{N} \tag{4}$$

in the case $A \geq \sum_{i=1}^k \beta_i$.
Let us first introduce some notation. Let us define the following sets of indices :
$I_\beta = \{i \in \{1,2,\ldots,k\} | \beta_i > 0\}$ and $I_B = \{j \in \{1,2,\ldots,k\} | B_j > 0\}$.
We will use these sets regularly when referring to the k^{th} order rational difference equation. We will also make regular use of the terms periodic solution, locally stable equilibrium point, and globally asymptotically stable equilibrium point. This terminology is defined in [1,p.4-5]. In [1,p.151], the authors mention that by making a change of variables we may assume that $gcd(I_\beta \cup I_B) = 1$. We shall make use of the same change of variables. For the convenience of the reader we shall include this change of variables in the following remark.

Remark 1. *If $gcd(I_\beta \cup I_B) > 1$ we may utilize the same change of variables as was used in* [1]*. Namely, given a solution $\{x_n\}_{n=0}^\infty$ we may introduce sequences $\{y_m^0\}_{m=0}^\infty$,...,$\{y_m^{\mu-1}\}_{m=0}^\infty$ where $\mu = gcd(I_\beta \cup I_B)$. For $L \in \{0,\ldots,\mu-1\}$, we define $y_m^L = x_{m\mu+L}$.*
Notice that each sequence $\{y_m^L\}_{m=0}^\infty$ satisfies the following difference equation,

$$y_m^L = \frac{\alpha + \sum_{i\in I_\beta} \beta_i y_{m-(i/\mu)}^L}{A + \sum_{j\in I_B} B_j y_{m-(j/\mu)}^L}, n \in \mathbb{N}. \tag{5}$$

Here our parameters $\beta_i, B_j, \alpha,$ and A are carried over from the original equation for all $i, j \in \{1, \dots, k\}$. It behooves us to write this in a more familiar format. Let $\beta_i^* = \beta_{i\mu}$ for $i \in \{1, \dots, k\}$, $i \leq (k/\mu)$ and let $\beta_i^* = 0$ for $i > (k/\mu)$. Similarly let $B_i^* = B_{i\mu}$ for $i \in \{1, \dots, k\}$, $i \leq (k/\mu)$ and let $B_i^* = 0$ for $i > (k/\mu)$. Using this new notation $\{y_m^L\}_{m=0}^{\infty}$ is a solution to the following difference equation,

$$z_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i^* z_{n-i}}{A + \sum_{j=1}^{k} B_j^* z_{n-j}}, n \in \mathbb{N}. \tag{6}$$

Where $\beta_i^*, B_j^* \geq 0$ for all $i, j \in \{1, \dots, k\}$ and α, A are carried over from the original equation. Also if $A \geq \sum_{i=1}^{k} \beta_i$ in the original equation then $A \geq \sum_{i=1}^{k} \beta_i^*$. Furthermore for Equation (6), $gcd(I_\beta \cup I_B) = 1$. In this way we may decouple our solution $\{x_n\}_{n=0}^{\infty}$ and so henceforth we shall assume that $gcd(I_\beta \cup I_B) = 1$.

In this article, we only address the unsolved case, namely the case where
$I_\beta \cap I_B = \emptyset$, $A = \sum_{i=1}^{k} \beta_i > 0$, and $\alpha = 0$.
The resolved cases are contained in [1], with references, and will be mentioned briefly in the conclusion of this article. Further, using Remark 1, we may assume without loss of generality that $gcd(I_\beta \cup I_B) = 1$.
In the proof of Theorem 3 we will make use of the Frobenius number. We introduce this concept in the following remark, for more details regarding the Frobenius number see [7].

Remark 2. *Given a set $I \subset \mathbb{N}$ with $|I| < \infty$ and $\gcd(I) = 1$, there exists $N_I \in \mathbb{N}$, called the Frobenius number of I, so that for $n \in \mathbb{N}$ with $n > N_I$, $n = \sum_{m=1}^{|I|} a_m i_m$ where a_m is a non-negative integer, and $i_m \in I$ for each $m \in \{1, \dots, |I|\}$. In particular, for the set of finite linear combinations of elements from I, $\ell(I) = \{\sum_{m=1}^{\nu} i_m | \nu \in \mathbb{N}$ and $i_m \in I$ for all $m \in \mathbb{N}\}$, $n \in \ell(I)$.*
A quick consequence of this result is that for $I \subset \mathbb{N}$ with $|I| < \infty$, then there exists $\eta_N \in \mathbb{N}$ so that for all $n \geq \eta_N$, $gcd(I)n \in \ell(I)$.

There is a natural way to consider k^{th} order difference equations as recursive equations on $\mathbb{R}^k$. Here we shall take the opportunity to introduce notation so that we are prepared for this transition. When we have $x \in [0, \infty)^k$ we may represent x as a k-tuple, $x = (x^1, \dots, x^k)$.

Theorem 3. *Consider the k^{th} order rational difference equation with non-negative initial contitions,*

$$y_n = \frac{\sum_{i=1}^{k} \beta_i y_{n-i}}{A + \sum_{j=1}^{k} B_j y_{n-j}}, n \in \mathbb{N}. \tag{7}$$

Let $\beta_i, B_j \geq 0$ for all $i, j \in \{0 \dots k\}$ and let $A = \sum_{i=1}^{k} \beta_i > 0$. Let $I_\beta \cap I_B = \emptyset$ and let $gcd(I_\beta \cup I_B) = 1$.
Then we have the following.

1. *If $gcd(I_\beta) | j$ for some $j \in I_B$ then the zero equilibrium is globally asymptotically stable.*

(2) *If there does not exist* $j \in I_B$ *so that* $gcd(I_\beta)|j$ *then every solution converges to a periodic solution of period* $gcd(I_\beta)$.

Proof. Observe that,

$$y_n = \frac{\sum_{i=1}^{k}\beta_i y_{n-i}}{A+\sum_{j=1}^{k}B_j y_{n-j}} \leq \frac{(\sum_{i=1}^{k}\beta_i) max_{i\in\{1,\dots,k\}} y_{n-i}}{A} \leq max_{i\in\{1,\dots,k\}} y_{n-i}. \tag{8}$$

This was noted in [1]. This implies that every solution of Equation (7) is bounded. Using our difference equation let us construct another recursive equation,

$$x_n = f(x_{n-1}), n \in \mathbb{N}, x_0 \in [0,\infty)^k, \tag{9}$$

where $f : [0,\infty)^k \to [0,\infty)^k$ and

$$f(x) = f((x^1,\dots,x^k)) = (\frac{\sum_{i=1}^{k}\beta_i x^i}{A+\sum_{j=1}^{k}B_j x^j}, x^1, x^2, \dots, x^{k-1}).$$

Further define $h : [0,\infty)^k \to [0,\infty)$ so that $h(x) = h((x^1,\dots,x^k)) = max_{i\in\{1,\dots,k\}} x^i$. In this case the metric on $[0,\infty)^k$ is the normal euclidean metric, and it is obvious that both f and h are continuous.
Continuing our argument, given $x \in [0,\infty)^k$,

$$h\circ f(x) = max(\frac{\sum_{i=1}^{k}\beta_i x^i}{A+\sum_{j=1}^{k}B_j x^j}, x^1, x^2, \dots, x^{k-1}) = max(\frac{\sum_{i=1}^{k}\beta_i x^i}{A+\sum_{j=1}^{k}B_j x^j}, max_{i\in\{1,\dots,k-1\}} x^i).$$

Since

$$\frac{\sum_{i=1}^{k}\beta_i x^i}{A+\sum_{j=1}^{k}B_j x^j} \leq \frac{(\sum_{i=1}^{k}\beta_i) max_{i\in\{1,\dots,k\}} x^i}{A} \leq max_{i\in\{1,\dots,k\}} x^i,$$

and
$max_{i\in\{1,\dots,k-1\}} x^i \leq max_{i\in\{1,\dots,k\}} x^i$ it follows that $h\circ f(x) \leq max_{i\in\{1,\dots,k\}} x^i = h(x)$.
Since every subsequence of a solution is a bounded sequence in $\mathbb{R}^k$, every subsequence of a solution has a convergent subsequence. Let us investigate the set
$\Omega = \{x \in [0,\infty)^k | h\circ f^j(x) = h(x) \text{ for all } j \in \mathbb{N}\}$.
We claim that $\Omega \subset \{x \in [0,\infty)^k | x^i \in \{0, h(x)\} \text{ for all } i \in \{1,\dots,k\}\}$.
Suppose $x_0 \notin \{x \in [0,\infty)^k | x^i \in \{0, h(x)\} \text{ for all } i \in \{1,\dots,k\}\}$, then $x_0^L \notin \{0, h(x_0)\}$ for some $L \in \{1,\dots,k\}$.
Notice that if $x_{n+1} = f(x_n), n \geq 0$, then $h(x_0) \geq h(x_1) \geq \dots \geq h(x_n) \geq h(x_{n+1}) \geq \dots$, and consequently $x_n^j \leq h(x_0)$ for all $j = 1,\dots,k$, and for all $n \geq 0$. Since $I_\beta \cap I_B = \emptyset$, if $x_n^\rho \notin \{0, h(x_0)\}$ for some $\rho \in I_\beta \cup I_B$, then either $\rho \in I_\beta$ and

$$0 \leq \frac{(\sum_{i=1}^{k}\beta_i x_n^i) - \beta_\rho x_n^\rho}{A+(\sum_{j=1}^{k}B_j x_n^j)} < x_{n+1}^1 < \frac{(\sum_{i=1}^{k}\beta_i x_n^i) + \beta_\rho(h(x_0) - x_n^\rho)}{A+(\sum_{j=1}^{k}B_j x_n^j)} \leq h(x_0). \tag{10}$$

Or, $\rho \in I_B$ and

$$h(x_0) \geq \frac{(\sum_{i=1}^{k} \beta_i x_n^i)}{A + (\sum_{j=1}^{k} B_j x_n^j) - B_\rho x_n^\rho} > x_{n+1}^1 > \frac{(\sum_{i=1}^{k} \beta_i x_n^i)}{A + (\sum_{j=1}^{k} B_j x_n^j) + B_\rho (h(x_0) - x_n^\rho)} \geq 0. \tag{11}$$

Hence, if $x_n^\rho \notin \{0, h(x_0)\}$ for some $\rho \in I_\beta \cup I_B$, then $x_{n+1}^1 \notin \{0, h(x_0)\}$. Since our original equation is a k^{th} order difference equation $k \in I_\beta \cup I_B$. Hence $x_{k-L}^k = x_0^L \notin \{0, h(x_0)\}$, so $x_{k-L+1}^1 \notin \{0, h(x_0)\}$.
Now suppose $x_n^1 \notin \{0, h(x_0)\}$, then $x_{n+s-1}^s = x_n^1 \notin \{0, h(x_0)\}$ for all $s \in I_\beta \cup I_B$. Hence $x_{n+s}^1 \notin \{0, h(x_0)\}$ for all $s \in I_\beta \cup I_B$. Using induction we see that this implies $x_{n+d}^1 \notin \{0, h(x_0)\}$ for any $d \in \ell(I_\beta \cup I_B) = \{\sum_{m=1}^{v} s_m | v \in \mathbb{N}$ and $s_m \in I_\beta \cup I_B$ for all $m \in \mathbb{N}\}$. Since $gcd(I_\beta \cup I_B) = 1$ using Remark 2, there exists $\eta \in \mathbb{N}$ so that if $n \geq \eta$ then $x_n^1 \notin \{0, h(x_0)\}$. Hence $h(x_{\eta+k+1}) \neq h(x_0)$ and so $x_0 \notin \Omega$.
We have shown that $\Omega \subset \{x \in [0,\infty)^k | x^i \in \{0, h(x)\}$ for all $i \in \{1, \dots, k\}\}$.
Hence $h^{-1}(\{b\}) \cap \Omega \subset h^{-1}(\{b\}) \cap \{x \in [0,\infty)^k | x^i \in \{0, h(x)\}$ for all $i \in \{1, \dots, k\}\} \subset \{x \in [0,\infty)^k | x^i \in \{0, b\}$ for all $i \in \{1, \dots, k\}\}$. So $|h^{-1}(\{b\}) \cap \Omega| < 2^k$ for all $b \in [0, \infty)$. Hence if $x_0 \in \Omega$, then x_0 is eventually periodic of period $p \leq 2^k$ for f.
We claim that if $x_0 \in \Omega$, then x_0 is eventually periodic of period $p = gcd(I_\beta)$.
Notice that $x_n^1 = 0$ only if $x_{n-1}^i = 0$ for all $i \in I_\beta$. However, this implies $x_{n-i}^1 = x_{n-1}^i = 0$ for all $i \in I_\beta$. Using induction we see that this implies $x_{n-D}^1 = 0$ for all $D \in \ell(I_\beta)$. Using Remark 2, there exists $\eta_D \in \mathbb{N}$ so that for $L_D \in \mathbb{N}$ where $n \geq gcd(I_\beta)L_D \geq \eta_D$ we have $x_{n-(gcd(I_\beta)L_D)}^1 = 0$.
Notice that $x_n^1 = h(x_0)$ only if $x_{n-1}^i = h(x_0)$ for all $i \in I_\beta$. However, this implies $x_{n-i}^1 = x_{n-1}^i = h(x_0)$ for all $i \in I_\beta$. Using induction we see that this implies $x_{n-D}^1 = h(x_0)$ for all $D \in \ell(I_\beta)$. Using Remark 2, there exists $\eta_D \in \mathbb{N}$ so that for $L_D \in \mathbb{N}$ where $n \geq gcd(I_\beta)L_D \geq \eta_D$ we have $x_{n-(gcd(I_\beta)L_D)}^1 = h(x_0)$.
Let $x_0 \in \Omega$, and suppose $x_n \neq x_{n-gcd(I_\beta)}$ for some $n \geq \eta_D + 3k + 1$. Then $x_n^i \neq x_{n-gcd(I_\beta)}^i$ for some $i \in \{1, \dots, k\}$, hence there exists $N_1 = n - (i-1) > \eta_D + 2k$ so that $x_{N_1}^1 \neq x_{N_1 - gcd(I_\beta)}^1$. Since $N_1 > \eta_D + 2k$ there exists $L_D \in \mathbb{N}$ so that $N_1 \geq (1 + L_D)gcd(I_\beta) > L_D gcd(I_\beta) \geq \eta_D$. But then $x_{N_1-(1+L_D)gcd(I_\beta)}^1 = x_{N_1}^1 \neq x_{N_1-gcd(I_\beta)}^1 = x_{N_1-(1+L_D)gcd(I_\beta)}^1$. This however, is clearly a contradiction. Hence if $x_0 \in \Omega$, then x_0 is eventually periodic with not necessarily prime period $gcd(I_\beta)$.

Further notice that $x_n^1 = h(x_0)$ only if $x_{n-1}^j = 0$ for all $j \in I_B$. However, this implies $x_{n-j}^1 = x_{n-1}^j = 0$ for all $j \in I_B$.
Now assume that $x_0 \in \Omega$ and $gcd(I_\beta)|j$ for some $j \in I_B$. Since $x_0 \in \Omega$ either $x_n^1 = h(x_0)$ for some $n \geq \eta_D + 4k + 1$, or $x_n^1 = 0$ for all $n \geq \eta_D + 4k + 1$. Assume $x_n^1 = h(x_0)$ for some $n \geq \eta_D + 4k + 1$ and choose $J \in I_B$ so that $J = gcd(I_\beta)\nu$ for some $\nu \in \mathbb{N}$. Then since $x_n^1 = h(x_0)$, $x_{n-J}^1 = 0$. However also $x_{n-J}^1 = x_{n-gcd(I_\beta)\nu}^1 = x_n^1 = h(x_0)$, since x_0 is eventually periodic with period $gcd(I_\beta)$. Thus $h(x_0) = 0$ whenever $x_n^1 = h(x_0)$ for some $n \geq \eta_D + 4k + 1$. If $x_n^1 = 0$ for all $n \geq \eta_D + 4k + 1$, then $h(x_0) = h(x_{\eta_D + 5k + 1}) = 0$. Hence $h(x_0) = 0$ for all $x_0 \in \Omega$ in Case 1.

Let us consider Case 2. In this case choose $x_0 = (x_0^1, \dots, x_0^k)$ where $x_0^i = b > 0$ if $gcd(I_\beta)|i$ and $x_0^i = 0$ otherwise for $i \in \{1, \dots, k\}$. Since there does not exist $j \in I_B$ so that $gcd(I_\beta)|j$, it is a simple matter to see that this is a non-zero periodic solution of prime period $gcd(I_\beta)$. Hence in Case 2 we must be content to show that every solution converges to a periodic solution of period $gcd(I_\beta)$.

Applying Theorem 2 in Case 1, we find that every solution of Equation (9) converges to $(0, \dots, 0)$. Applying Theorem 2 in Case 2, we find that every solution of Equation (9) converges to a periodic solution of period $gcd(I_\beta)$.

Letting $x_0 = (x_0^1, \dots, x_0^k)$ be determined by our initial conditions according to $x_0^i = y_{1-i}$, we find, using induction, that $x_n^1 = y_n$. Hence in Case 1, we find that every solution of Equation (7) converges to the zero equilibrium. As was mentioned in [1], it is self-evident that the zero equilibrium is locally stable for both of these cases. Therefore, in Case 1, the zero equilibrium is globally asymptotically stable.

Furthermore in Case 2 it is clear that every solution of Equation (7) converges to a periodic solution of period $gcd(I_\beta)$.

□

4. Conclusion

Let us summarize the behavior of solutions of the k^{th} order rational difference equation, with non-negative parameters and non-negative initial conditions,

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, n \in \mathbb{N} \tag{12}$$

in the case $A \geq \sum_{i=1}^{k} \beta_i$.

If $A > \sum_{i=1}^{k} \beta_i$, then the unique equilibrium of Equation (12) is globally asymptotically stable. This follows from Theorem 5.17.1 in [1] when $\sum_{i=1}^{k} \beta_i = 0$, and from Theorem 5.23.2 in [1] otherwise.

If $A = \sum_{i=1}^{k} \beta_i = 0$ and $gcd(I_B) = gcd(\{i + j | i, j \in I_B\})$, then the unique equilibrium of Equation (12) is globally asymptotically stable. However when $A = \sum_{i=1}^{k} \beta_i = 0$ and $gcd(I_B) \neq gcd(\{i + j | i, j \in I_B\})$ then every solution of Equation (12) converges to a periodic solution of period $2gcd(I_B)$. This follows from theorem 5.17.2 in [1].

We have shown above that whenever $I_\beta \cap I_B = \emptyset$, $gcd(I_\beta \cup I_B) = 1$, $A = \sum_{i=1}^{k} \beta_i > 0$, and $\alpha = 0$, the following holds.

(1) If $gcd(I_\beta)|j$ for some $j \in I_B$ then the unique equilibrium of Equation (12) is globally asymptotically stable.
(2) If there does not exist $j \in I_B$ so that $gcd(I_\beta)|j$ then every solution of Equation (12) converges to a periodic solution of period $gcd(I_\beta)$.

We shall show that the conditions $I_\beta \cap I_B = \emptyset$ and $gcd(I_\beta \cup I_B) = 1$ are superfluous. First we shall remove the assumption $gcd(I_\beta \cup I_B) = 1$. Assume $I_\beta \cap I_B = \emptyset$, $gcd(I_\beta \cup I_B) \neq 1$, $A = \sum_{i=1}^{k} \beta_i > 0$, and $\alpha = 0$. We may now make use of the change of variables mentioned in Remark 1. Using this we may rewrite an arbitrary solution of our difference equation using the equation $x_{m\mu+L} = y^L_m$ where $L \in \{0, \ldots, \mu - 1\}$ and $\mu = gcd(I_\beta \cup I_B)$. Recall that the sequence $\{y^L_m\}$ is a solution of Equation (6). So that no confusion arises, we label the sets of indices for Equation (6) I^*_β and I^*_B and reserve the usual notation for our original difference equation. It is a simple matter to compute that $I^*_\beta = \{\frac{i}{gcd(I_\beta \cup I_B)} | i \in I_\beta\}$ and $I^*_B = \{\frac{i}{gcd(I_\beta \cup I_B)} | i \in I_B\}$. Hence $gcd(I^*_\beta \cup I^*_B) = 1$ and so our result applies to Equation (6).

Suppose $gcd(I_\beta)|j$ for some $j \in I_B$, then $gcd(I^*_\beta)|j$ for some $j \in I^*_B$. In this case every solution of Equation (6) converges to the unique equilibrium. A simple computation yields that our original difference equation and Equation (6) have the same equilibrium. Hence in this case every solution to our original difference equation converges to the unique equilibrium.

Suppose there does not exist $j \in I_B$ so that $gcd(I_\beta)|j$, then there does not exist $j \in I^*_B$ so that $gcd(I^*_\beta)|j$. In this case every solution of Equation (6) converges to a periodic solution of period $gcd(I^*_\beta)$. Notice that $x_n - x_{n+gcd(I_\beta)} = x_{m\mu+L} - x_{m\mu+L+gcd(I_\beta)} = x_{m\mu+L} - x_{(m+gcd(I^*_\beta))\mu+L}$ since $\mu = gcd(I_\beta \cup I_B)$. So $x_n - x_{n+gcd(I_\beta)} = y^L_m - y^L_{m+gcd(I^*_\beta)}$. Hence in this case every solution to our original difference equation converges to a periodic solution of not necessarily prime period $gcd(I_\beta)$.

We have successfully removed the assumption $gcd(I_\beta \cup I_B) = 1$. Let us now suppose $I_\beta \cap I_B \neq \emptyset$, $A = \sum_{i=1}^k \beta_i > 0$, and $\alpha = 0$. Clearly $gcd(I_\beta)|j$ for $j \in I_\beta \cap I_B$. Thus we want to show that the unique equilibrium of Equation (12) is globally asymptotically stable in this case. This is precisely Theorem 5.23.4 part (b) in [1]. Hence whenever $A = \sum_{i=1}^k \beta_i > 0$, and $\alpha = 0$, the following holds.

1. If $gcd(I_\beta)|j$ for some $j \in I_B$ then the unique equilibrium of Equation (12) is globally asymptotically stable.
2. If there does not exist $j \in I_B$ so that $gcd(I_\beta)|j$ then every solution of Equation (12) converges to a periodic solution of period $gcd(I_\beta)$.

Suppose $A = \sum_{i=1}^k \beta_i > 0$, $\alpha > 0$, and $gcd(I_\beta \cup I_B) = 1$, then the following holds.

1. If i is even for all $i \in I_\beta$ and j is odd for all $j \in I_B$, then every solution of Equation (12) converges to a periodic solution of period 2.
2. If not, then every solution of Equation (12) converges to the unique equilibrium.

If $A = \sum_{i=1}^k \beta_i > 0$, $\alpha > 0$, and $gcd(I_\beta \cup I_B) \neq 1$ then we must make a change of variables and apply the prior result. This is precisely theorem 5.23.3 in [1].

Finally, we note that the results in [1] are a compilation of the combined work of many authors. We encourage the reader to refer to the citation list in [1] for a full account. We list, for the convienience of the reader, several references used in [1] which are particularly relevant to this result, namely [2],[3],[4], and [5].

REFERENCES

[1] E. Camouzis and G. Ladas, *Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC Press, 2007.

[2] E. Camouzis, G. Ladas, F. Palladino, and E.P. Quinn, On the Boundedness Character of Rational Equations, Part 1, *J. Difference Equa. Appl.* **12**(2006), 503-523.

[3] H.A. El-Metwally, E.A. Grove, and G. Ladas, A global convergence result with applications to periodic solutions, *J. Math. Anal. Appl.* **245**(2000), 161-170.

[4] H.A. El-Metwally, E.A. Grove, G. Ladas, and H.D. Voulov, On the global attractivity and the periodic character of some difference equations, *J. Difference Equa. Appl.* **7**(2001), 837-850.

[5] E.A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman & Hall/CRC Press,2005.

[6] F.J. Palladino, Difference inequalities, comparison tests, and some consequences, *Involve J. Math.* **1**(2008), 101-110.

[7] J.L. Ramírez Alfonsín, *The Diophantine Frobenius problem. Oxford Lecture Series in Mathematics and its Applications 30*, Oxford University Press, Oxford, 2005.