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Open problems and conjectures

Edited by Gerry Ladas

In this section we present some open problems and conjectures about some interesting types of difference equations. Please submit your problems and conjectures with all relevant information to G. Ladas: Email: gladas@math.uri.edu

I pledge to donate the amount of $1,000 (USA) to the International Society of Difference Equations, provided that the complete solutions of the open problems and conjectures in this paper are brought to the attention of myself and the President of the Society by the end of the year 2006.

On the boundedness character of rational equations, part 1

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Our goal here is to present some new results and to pose several open problems and conjectures on the boundedness character of solutions of some special cases of the k^{th}-order rational difference equation

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{i=1}^{k} B_i x_{n-i}}, n = 1, 2, \ldots \tag{1}$$

with nonnegative parameters and with arbitrary nonnegative initial conditions such that the denominator is always positive. For related work, see [1–10].

Using the numbering system which was introduced in reference [10], the only conjectures on boundedness which were posed in [6] and which have not yet been confirmed or refuted are about the following 11 equations. See [1] and [3]:

#28, #44, #56, #59, #70, #88, #94, #120, #123, #170, #176

Here, we will first confirm the boundedness character which has been conjectured in [6] for the special cases

#88 and #170.

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THEOREM 1 *Every solution of each of the following two equations with positive parameters and positive initial conditions is bounded:*

$$\#88: \quad x_{n+1} = \frac{\beta x_n + x_{n-1}}{x_{n-1} + Dx_{n-2}}, \quad n = 0, 1, \ldots. \tag{2}$$

$$\#170: \quad x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{x_{n-1} + Dx_{n-2}}, \quad n = 0, 1, \ldots. \tag{3}$$

Proof. We will give the proof for the special case #88. The proof for the special case #170 is similar and will be omitted. Assume for the sake of contradiction that equation (2) has a positive unbounded solution $\{x_n\}$. Then there exists a subsequence $\{n_i\}$ such that

$$x_{n_i+1} \to \infty \tag{4}$$

and for every i,

$$x_{n_i+1} > x_j, \quad \text{for} \quad \text{all} \quad j < n_i + 1. \tag{5}$$

Then clearly,

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} + Dx_{n-2}} + \beta \frac{\beta x_{n-1} + x_{n-2}}{x_{n-1} + Dx_{n-2}} \, \frac{1}{x_{n-2} + Dx_{n-3}},$$

from which it follows that

$$x_{n_i-2} \to 0, \qquad x_{n_i-3} \to 0, \tag{6}$$

$$x_{n_i-5} + Dx_{n_i-6} \to \infty, \quad \text{and} \qquad x_{n_i-6} + Dx_{n_i-7} \to \infty. \tag{7}$$

Now we claim that

$$\liminf_{i \to \infty} x_{n_i-1} > 0. \tag{8}$$

Otherwise, there exists a subsequence of $\{n_i\}$, which for economy of notation we still denote by $\{n_i\}$, such that

$$x_{n_i-1} \to 0. \tag{9}$$

Therefore,

$$x_{n_i-4} + Dx_{n_i-5} \to \infty. \tag{10}$$

Also,

$$x_{n_i+1} \leq 1 + \beta \frac{\max(\beta, 1)}{\min(1, D)} \frac{1}{Dx_{n-3}} = 1 + \frac{\beta}{D} \frac{\max(\beta, 1)}{\min(1, D)} \, \frac{x_{n_i-5} + Dx_{n_i-6}}{\beta x_{n_i-4} + x_{n_i-5}}$$

and so

$$x_{n_i+1} \leq 1 + M\left(1 + \frac{Dx_{n_i-6}}{\beta x_{n_i-4} + x_{n_i-5}}\right)$$

where

$$M = \frac{\beta \max(\beta, 1)}{D \min(1, D)}.$$

Hence eventually,

$$x_{n_i+1} < x_{n_i-6}$$

which contradicts (5) and establishes our claim that (8) holds.

Observe that

$$x_{n_i-1} = \frac{\beta x_{n_i-2} + x_{n_i-3}}{x_{n_i-3} + D x_{n_i-4}} = \frac{\beta\left(\frac{x_{n_i-2}}{x_{n_i-3}}\right) + 1}{1 + D\left(\frac{x_{n_i-4}}{x_{n_i-3}}\right)}$$

and

$$x_{n_i-1} = \frac{\beta x_{n_i-2} + x_{n_i-3}}{x_{n_i-3} + D x_{n_i-4}} = \frac{\beta + \frac{x_{n_i-3}}{x_{n_i-2}}}{\frac{x_{n_i-3}}{x_{n_i-2}} + D\left(\frac{x_{n_i-4}}{x_{n_i-2}}\right)}$$

and so the following two statements are true:

If the sequence

$$\left\{\frac{x_{n_i-2}}{x_{n_i-3}}\right\} \quad \text{is} \quad \text{bounded}$$

then

$$\liminf_{i \to \infty} \left(\frac{x_{n_i-3}}{x_{n_i-4}}\right) > 0,$$

and if the sequence

$$\left\{\frac{x_{n_i-2}}{x_{n_i-3}}\right\} \quad \text{is} \quad \text{unbounded,}$$

then

$$\liminf_{i \to \infty} \left(\frac{x_{n_i-2}}{x_{n_i-4}}\right) > 0.$$

Hence, there exists $\mu > 0$ such that eventually

$$\text{either}\, x_{n_i-3} > \mu x_{n_i-4} \;\text{ or }\; x_{n_i-2} > \mu x_{n_i-4}.$$

Then there exists a positive number K such that

$$x_{n_i+1} \le 1 + \beta \frac{\max(\beta, 1)}{\min(D, 1)} \frac{1}{x_{n_i-2} + D x_{n_i-3}} < 1 + K \frac{1}{x_{n_i-4}}$$

$$= 1 + K \frac{x_{n_i-6} + D x_{n_i-7}}{\beta x_{n_i-5} + x_{n_i-6}} \le 1 + K\left(1 + \frac{D x_{n_i-7}}{\beta x_{n_i-5} + x_{n_i-6}}\right)$$

Therefore eventually,

$$x_{n_i+1} < x_{n_i-7}$$

which contradicts (5) and completes the proof. ☐

CONJECTURE 1 (*a*) *Show that each of the following seven equations has unbounded solutions in some range of its parameters:*

$$\#28, \quad \#44, \quad \#56, \quad \#59, \quad \#70, \quad \#120, \quad \#123\,.$$

(*b*) *Show that every solution of each of the two equations,*

$$\#94 \quad \text{and} \quad \#176$$

is bounded.

CONJECTURE 2 *Assume that either*

$$D \leq 1$$

or that

$$D > 1 \quad \text{and} \quad \frac{D-1}{D+3} < \beta < \frac{D^2+3D+1}{D^2}.$$

Then every solution of equation (2) *converges to the equilibrium.*

CONJECTURE 3 *Assume that*

$$D > 1 \quad \text{and} \quad \beta < \frac{\sqrt{D}-1}{\sqrt{D}+3}.$$

Then equation (2) *has the unique prime period-two solution*

$$\ldots, \phi, \psi, \phi, \psi, \ldots \tag{11}$$

where ϕ *and* ψ *are the positive roots of the quadratic equation*

$$(D-1)t^2 + (\beta-1)(D-1)t + \beta = 0,$$

and (11) *is locally asymptotically stable.*

The following result established the boundedness of every solution in

$$4 \cdot 3^k - 2^{k+1} - 1$$

special cases of equation (1).

THEOREM 2 *Assume that for every* $i \in \{1, \ldots, k\}$ *for which the parameter* β_i *in the numerator of equation* (1) *is positive, the corresponding parameter* B_i *in the denominator of the equation is also positive. Then every solution of equation* (1) *is bounded from above by a positive number.*

Proof. Let us denote by I_β and I_0 the following subsets of $\{1,\ldots,k\}$:

$$\begin{aligned} I_\beta &= \{i \in \{1,\ldots,k\} : \beta_i > 0\} \\ I_0 &= \{i \in \{1,\ldots,k\} : \beta_i = 0 \quad \text{and} \quad B_i > 0\}. \end{aligned}$$

Then in view of the hypothesis that $\beta_i > 0 \Rightarrow B_i > 0$, equation (1) becomes,

$$x_n = \frac{\alpha + \sum_{i\in I_\beta} \beta_i x_{n-i}}{A + \sum_{i\in I_\beta} B_i x_{n-i} + \sum_{i\in I_0} B_i x_{n-i}}, \quad n = 1,2,\ldots$$

Case 1. $A > 0$.

Then

$$x_n \le \frac{\max_{i\in I_\beta}(\alpha,\beta_i)\Big(1+\sum_{i\in I_\beta} x_{n-i}\Big)}{\min_{i\in I_\beta}(A,B_i)\Big(1+\sum_{i\in I_\beta} x_{n-i}\Big)} = \frac{\max_{i\in I_\beta}(\alpha,\beta_i)}{\min_{i\in I_\beta}(A,B_i)}$$

and the solution is bounded.

Case 2. $\alpha = 0$.

Then

$$x_n \le \frac{\big(\max_{i\in I_\beta}\beta_i\big)\sum_{i\in I_\beta} x_{n-i}}{\big(\min_{i\in I_\beta}B_i\big)\sum_{i\in I_\beta} x_{n-i}} = \frac{\max_{i\in I_\beta}\beta_i}{\min_{i\in I_\beta}B_i}$$

and the solution is bounded.

Case 3. $A = 0, \quad \alpha > 0$, and $I_0 = \emptyset$.

Then

$$x_n = \frac{\alpha + \sum_{i\in I_\beta} \beta_i x_{n-i}}{\sum_{i\in I_\beta} B_i x_{n-i}} > \frac{\min_{i\in I_\beta}\beta_i}{\max_{i\in I_\beta}B_i}$$

and the solution is bounded from below by the positive number

$$L = \frac{\min_{i\in I_\beta}\beta_i}{\max_{i\in I_\beta}B_i}.$$

But then the solution is also bounded from above, because,

$$x_n \le \frac{\alpha}{L\sum_{i\in I_\beta} B_i} + \frac{\max_{i\in I_\beta}\beta_i}{\min_{i\in I_\beta}B_i}.$$

Case 4. $A = 0, \quad \alpha > 0$, and $I_\beta = \emptyset$.

In this case,

$$x_{n+1}=\frac{\alpha}{\sum_{i\in I_0}B_ix_{n-i}}.$$

Choose positive numbers L and U such that

$$x_1,\ldots,x_k\in[L,U]$$

and

$$LU=\frac{\alpha}{\sum_{i\in I_0}B_i}.$$

Then

$$L=\frac{\alpha}{\left(\sum_{i\in I_0}B_i\right)U}\leq x_{k+1}=\frac{\alpha}{\sum_{i\in I_0}B_ix_{k+1-i}}\leq\frac{\alpha}{\left(\sum_{i\in I_0}B_i\right)L}=U$$

and by induction

$$x_n\in[L,U],\quad\text{for}\quad n\geq 1.$$

Case 5. $A=0,\quad\alpha>0,\quad I_\beta\neq\emptyset$, and $I_0\neq\emptyset$.

Choose positive numbers L and U such that

$$x_1,\ldots,x_k\in[L,U]$$

and

$$LU=\frac{\alpha}{\sum_{i\in I_0}B_i}.$$

with

$$L<\min_{i\in I_\beta}\left\{\frac{\beta_i}{B_i},\left(\frac{\alpha}{\sum_{i\in I_0}B_i}\right)\frac{B_i}{\beta_i}\right\}.$$

Then one can see by induction that

$$x_n\in[L,U],\quad\text{for}\,n\geq 1.$$

and the proof is complete. □

Theorem 2 is a remarkable result which establishes the boundedness of every solution in 91 special cases of equation (1), when $k=3$, and in 291 special cases of equation (1), when $k=4$.

The 91 special cases of equation (1) when $k = 3$, where every solution of the equation is bounded are the following:

#1 − 4, #6, #11, #16 − 23, #26,

#27, #30, #32, #34, #37,

#39, #40, #42, #47, #52,

#65, #68, #69, #72, #74,

#76, #79, #81, #82, #86,

#93, #100 − 106, #108, #109, #111,

#112, #114 − 116, #133 − 136, #141, #142,

#114, #145, #147, #148 #150 − 153,

#156, #158, #160 #163, #164,

#168, #175, #182, #189 − 194, #201,

#204, #206, #208, #221, #212,

#216 − 220, #224, #225

From the above list of 91 special cases of equation (1), it is interesting to determine those whose behavior is "complicated" in the sense that in some region of the parameters there exist solutions of the equation which do not converge to an equilibrium point or to a periodic solution.

For example, each of the following 32 equations seems to have "complicated behavior" in the sense that in some region of the parameters there exist solutions of the equation which do not converge to an equilibrium point or to a periodic solution:

#27, #39, #40, #69, #81,

#82, #93, #100, #106, #108,

#114, #115, #116, #136, #142,

#156, #158, #160, #163, #164,

#175, #182, #189, #191, #192,

#193, #194, #204, #206, #208,

#211, #212,

Open Problem 1 *For each of the* 32 *equations in the above list, determine whether it has complicated behavior.*

In contrast to the above behavior of third order rational equations, we offer the following conjecture for second order rational equations.

CONJECTURE 4 *For the second order rational difference equation*

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}, \quad n = 0, 1, \ldots$$

with nonnegative parameters and nonnegative initial conditions such that

$$\gamma + A + B > 0,$$

show that every bounded solution of the equation converges to a (*not necessarily prime*) *period-two solution of the equation.*

When we deal with values of $k \geq 4$, it will be convenient for us to switch to a new notation for the special cases of equation (1) defined as follows:

Using the parameters in equation (1), set

$$v_i = \begin{cases} 2^i & \text{if } \beta_i > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad V_i = \begin{cases} 2^{i+1} & \text{if } B_i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the identifying number assigned to a special case of equation (1) is

$$\sum_{i=1}^{k} (v_i + V_i).$$

In the sequel, we will use the above numbering system for equations of order ≥ 4. For equations of order ≤ 3, we still use the notation which was introduced in [10].

Theorem 1 establishes the boundedness of every solution in 291 special cases of the equation

$$x_n = \frac{\alpha + \sum_{i=1}^{4} \beta_i x_{n-i}}{A + \sum_{i=1}^{4} B_i x_{n-i}}, \quad n = 1, 2, \ldots, \tag{12}$$

and 200 of these cases are fourth-order equations. They are the special cases of equation (12) with the following numbers:

#513, #515, #521, #523 − 527, #545,
#547, #553, #555 − 563, #568 − 575, #641,
#643, #649, #651 − 655, #673, #675,
#681, #683 − 691, #696 − 707, #712 − 719, #736 − 739,
#744 − 755, #760 − 771, #776 − 783, #800 − 803, #808 − 819,
#824 − 831, #896 − 899, #904 − 911, #928 − 931, #936 − 947,
#952 − 963, #968 − 975, #992 − 995, #1000 − 1011, #1016 − 1023.

In the above list of 200 fourth-order rational difference equations, it is interesting to determine all special cases with "complicated" behavior in the sense that in some region of the parameters there exist solutions of the equation which do not converge to an equilibrium point or to a periodic solution.

For example, each of the following 144 equations seems to have "complicated behavior" in the sense that in some region of the parameters there exist solutions of the equation which do not converge to an equilibrium point or to a periodic solution.

$$\begin{array}{lllll}
\#524-527, & \#556-559, & \#568, & \#572-575, & \#652-655, \\
\#684-687, & \#700-707, & \#712-719, & \#736-739, & \#744-755, \\
\#760-767, & \#780-783, & \#808-815, & \#824-831, & \#908-911, \\
\#928-931, & \#936-947, & \#952-963, & \#968-975, & \#992-995, \\
\#1000-1011, & \#1016-1022. & & &
\end{array}$$

On the other hand, none of the following 16 equations seems to have complicated behavior:

#689, #690, #691, #697, #699,
#776, #777, #778, #779, #896,
#897, #898, #899, #905, #906,
#907.

Open Problem 2 *Establish the global character of solutions of each of the* 16 *special cases in the above list.*

The next three theorems, which were presented in [3], established boundedness by the method of iteration in the following 16 special cases of equation (1) with $k = 3$:

#25, #60, #67, #91, #107, #124, #143, #155,
#159, #173, #188, #200, #203, #207, #215, #223

CONJECTURE 5 *Show that none of the* 16 *equations in the above list has the property that every solution of the equation converges to either an equilibrium point or a periodic solution.*

THEOREM 3 *Assume that*

$$\beta, A, C, D \in (0, \infty) \quad \textit{and} \quad \alpha, \gamma, \delta \in [0, \infty).$$

Then every nonnegative solution of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, \ldots \tag{13}$$

is bounded.

THEOREM 4 *Assume that*

$$\gamma, \delta, C, D \in (0, \infty) \quad \textit{and} \quad \alpha, \beta \in [0, \infty).$$

Then every nonnegative solution of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, \ldots \tag{14}$$

is bounded.

THEOREM 5 *Assume that*

$$\beta, \delta + A \in (0, \infty) \quad \textit{and} \quad \alpha, \delta, A \in [0, \infty).$$

Then every nonnegative solution of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2}}{A + x_{n-2}}, \quad n = 0, 1, \ldots \tag{15}$$

is bounded.

The proof of each of the above three theorems follows from by the so-called "method of iteration". That is, by observing that when we write x_{n+2} or x_{n+3} in terms of x_n, x_{n-1}, and x_{n-2}, every solution of the resulting equation is bounded.

Boundedness by iteration can be easily extended to higher order rational difference equations. For example, it can be used to establish the boundedness of all solutions in each of the following 98, fourth-order, special cases of equation (1) with $k = 4$:

#518, #519, #534, #535, #538,
#539, #542, #543, #550, #551,
#566, #567, #646, #647, #658,
#659, #662, #663, #666, #667,
#670, #671, #678, #679, #694,
#695, #710, #711, #722, #723,
#726, #727, #730, #731, #734,
#735, #742, #743, #758, #759,
#772 − 775, #788 − 791, #794 − 799, #806, #807,
#820 − 823, #902, #903, #914, #915,
#918, #919, #922, #923, #926,
#927, #934, #935, #950, #951,
#964 − 967, #976 − 983, #986 − 991, #998, #999,
#1012 − 1015.

Open Problem 3 *From the above list of* 98 *fourth-order rational difference equations, determine whether there are any whose equilibrium is globally asymptotically stable.*

Before we state the next result for equation (1), we introduce the following notation:

$$I_\beta = \left\{ j \in \{1, 2, \ldots, k\} : \beta_j > 0 \right\}$$

$$I_B = \left\{ j \in \{1, 2, \ldots, k\} : B_j > 0 \right\}$$

THEOREM 6 *Assume that either*

$$A > 0$$

or

$$A = 0, \quad I_B \neq \emptyset, \quad \textit{and} \quad I_B \subset I_\beta.$$

Furthermore, assume that for every sequence

$$\{c_m\}_{n=1}^{\infty} \quad \textit{with} \quad c_m \in I_\beta,$$

there exist positive integers N_1 *and* N_2 *such that*

$$\left(\sum_{m=N_1}^{N_2} c_m\right) \quad \in I_B.$$

Then every solution of equation (1) *is bounded.*

The proof is tedious but straightforward and its details will be omitted.

Open Problem 4 *Assume that*

$$A \in [0, \infty) \quad \textit{and} \quad i, j, l, m \in \{1, 2, \ldots\}.$$

Obtain (*easily verifiable*) *necessary and sufficient conditions, in terms of A, i,j,l, and m such that every positive solution of the equation*

$$x_n = \frac{x_{n-i} + x_{n-j}}{A + x_{n-l} + x_{n-m}}, \quad n = 1, 2, \ldots$$

is bounded.

Extensive computer observations and a substantial number of analytic investigations have led us to the following conjecture:

CONJECTURE 6 *Equation* (12) *has* 542 *special cases where every solution of the equation is bounded and in each of the remaining* 419 *special cases the equation has unbounded solutions in some region of its parameters.*

In 135 special cases of equation (12) where every solution of the equation is bounded, the order of the equation is 3 or less, see [6]. In the remaining

$$542 - 135 = 407$$

special cases, the equation is of order 4.

We have already listed 298 special cases of fourth-order rational equations where every solution of the equation is bounded as a consequence of Theorem 1 or Theorem 2. Also, in the following 8 fourth-order special cases of equation (12),

$$\#528 - 531 \quad \text{and} \quad \#784 - 787$$

the equations are reducible to second-order rational equations where every solution of the equation is known to be bounded. See [6].

If our Conjecture 4 is true, then there remain only

$$542 - (135 + 200 + 98 + 8) = 101$$

special cases of equation (12), each of order four, for which we conjecture that every solution of the equation is bounded. They are the following 101 special cases:

$$\#299: \quad x_{n+1} = \frac{\alpha + \epsilon x_{n-3}}{A + Bx_n + Cx_{n-1}}$$

$$\#419: \quad x_{n+1} = \frac{\alpha + \epsilon x_{n-3}}{A + Cx_{n-1} + Dx_{n-2}}$$

$$\#422: \quad x_{n+1} = \frac{\beta x_n + \epsilon x_{n-3}}{A + Cx_{n-1} + Dx_{n-2}}$$

$$\#423: \quad x_{n+1} = \frac{\alpha + \beta x_n + \epsilon x_{n-3}}{A + Cx_{n-1} + Dx_{n-2}}$$

$$\#436: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \epsilon x_{n-3}}{Cx_{n-1} + Dx_{n-2}}$$

$$\#437: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \epsilon x_{n-3}}{Cx_{n-1} + Dx_{n-2}}$$

$$\#438: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \epsilon x_{n-3}}{A + Cx_{n-1} + Dx_{n-2}}$$

$$\#439: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \epsilon x_{n-3}}{A + Cx_{n-1} + Dx_{n-2}}$$

$$\#484: \quad x_{n+1} = \frac{\beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{Cx_{n-1} + Dx_{n-2}}$$

$$\#485: \quad x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{Cx_{n-1} + Dx_{n-2}}$$

$$\#486: \quad x_{n+1} = \frac{\beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{A + Cx_{n-1} + Dx_{n-2}}$$

$$\#487: \quad x_{n+1}=\frac{\alpha+\beta x_n+\delta x_{n-2}+\epsilon x_{n-3}}{A+Cx_{n-1}+Dx_{n-2}}$$

$$\#500: \quad x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}+\epsilon x_{n-3}}{Cx_{n-1}+Dx_{n-2}}$$

$$\#501: \quad x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}+\epsilon x_{n-3}}{Cx_{n-1}+Dx_{n-2}}$$

$$\#502: x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}+\epsilon x_{n-3}}{A+Cx_{n-1}+Dx_{n-2}}$$

$$\#503: \quad x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}+\epsilon x_{n-3}}{A+Cx_{n-1}+Dx_{n-2}}$$

$$\#536: \quad x_{n+1}=\frac{\gamma x_{n-1}}{Bx_n+Ex_{n-3}}$$

$$\#540: \quad x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}}{Bx_n+Ex_{n-3}}$$

$$\#541: \quad x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{Bx_n+Ex_{n-3}}$$

$$\#565: \quad x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{Cx_{n-1}+Ex_{n-3}}$$

$$\#579: \quad x_{n+1}=\frac{\alpha+\delta x_{n-2}}{A+Ex_{n-3}}$$

$$\#582: \quad x_{n+1}=\frac{\beta x_n+\delta x_{n-2}}{A+Ex_{n-3}}$$

$$\#583: \quad x_{n+1}=\frac{\alpha+\beta x_n+\delta x_{n-2}}{A+Ex_{n-3}}$$

$$\#587: \quad x_{n+1}=\frac{\alpha+\delta x_{n-2}}{A+Bx_n+Ex_{n-3}}$$

$$\#588: \quad x_{n+1} = \frac{\beta x_n + \delta x_{n-2}}{Bx_n + Ex_{n-3}}$$

$$\#589: \quad x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2}}{Bx_n + Ex_{n-3}}$$

$$\#590: \quad x_{n+1} = \frac{\beta x_n + \delta x_{n-2}}{A + Bx_n + Ex_{n-3}}$$

$$\#591: \quad x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2}}{A + Bx_n + Ex_{n-3}}$$

$$\#602: \quad x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Ex_{n-3}}$$

$$\#603: \quad x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Ex_{n-3}}$$

$$\#604: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{Bx_n + Ex_{n-3}}$$

$$\#605: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{Bx_n + Ex_{n-3}}$$

$$\#606: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Ex_{n-3}}$$

$$\#607: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Ex_{n-3}}$$

$$\#656: \quad x_{n+1} = \frac{\gamma x_{n-1}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#657: \quad x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#664: \quad x_{n+1} = \frac{\gamma x_{n-1}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#665: \quad x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#668: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#669: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#693: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{Cx_{n-1} + Dx_{n-2} + Ex_{n-3}}$$

$$\#708: \quad x_{n+1} = \frac{\beta x_n + \delta x_{n-2}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#709: \quad x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#720: \quad x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#721: \quad x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#724: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#725: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#728: \quad x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#729: \quad x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#732: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#733: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#740: \quad x_{n+1} = \frac{\beta x_n + \delta x_{n-2}}{Cx_{n-1} + Dx_{n-2} + Ex_{n-3}}$$

$$\#741: \quad x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2}}{Cx_{n-1} + Dx_{n-2} + Ex_{n-3}}$$

$$\#756: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{Cx_{n-1} + Dx_{n-2} + Ex_{n-3}}$$

$$\#757: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{Cx_{n-1} + Dx_{n-2} + Ex_{n-3}}$$

$$\#792: \quad x_{n+1} = \frac{\gamma x_{n-1} + \epsilon x_{n-3}}{Bx_n + Ex_{n-3}}$$

$$\#793: \quad x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \epsilon x_{n-3}}{Bx_n + Ex_{n-3}}$$

$$\#804: \quad x_{n+1} = \frac{\beta x_n + \epsilon x_{n-3}}{Cx_{n-1} + Ex_{n-3}}$$

$$\#805: \quad x_{n+1} = \frac{\alpha + \beta x_n + \epsilon x_{n-3}}{Cx_{n-1} + Ex_{n-3}}$$

$$\#832: \quad x_{n+1} = \frac{\delta x_{n-2} + \epsilon x_{n-3}}{E x_{n-3}}$$

$$\#833: \quad x_{n+1} = \frac{\alpha + \delta x_{n-2} + \epsilon x_{n-3}}{E x_{n-3}}$$

$$\#834: \quad x_{n+1} = \frac{\delta x_{n-2} + \epsilon x_{n-3}}{A + E x_{n-3}}$$

$$\#835: \quad x_{n+1} = \frac{\alpha + \delta x_{n-2} + \epsilon x_{n-3}}{A + E x_{n-3}}$$

$$\#836: \quad x_{n+1} = \frac{\beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{E x_{n-3}}$$

$$\#837: \quad x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{E x_{n-3}}$$

$$\#838: \quad x_{n+1} = \frac{\beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{A + E x_{n-3}}$$

$$\#839: \quad x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{A + E x_{n-3}}$$

$$\#840: \quad x_{n+1} = \frac{\delta x_{n-2} + \epsilon x_{n-3}}{B x_n + E x_{n-3}}$$

$$\#841: \quad x_{n+1} = \frac{\alpha + \delta x_{n-2} + \epsilon x_{n-3}}{B x_n + E x_{n-3}}$$

$$\#842: \quad x_{n+1} = \frac{\delta x_{n-2} + \epsilon x_{n-3}}{A + B x_n + E x_{n-3}}$$

$$\#843: \quad x_{n+1} = \frac{\alpha + \delta x_{n-2} + \epsilon x_{n-3}}{A + B x_n + E x_{n-3}}$$

$$\#844: \quad x_{n+1} = \frac{\beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{B x_n + E x_{n-3}}$$

$$\#845: \quad x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{B x_n + E x_{n-3}}$$

$$\#846: \quad x_{n+1} = \frac{\beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{A + B x_n + E x_{n-3}}$$

$$\#847: \quad x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{A + B x_n + E x_{n-3}}$$

$$\#856: \quad x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3}}{B x_n + E x_{n-3}}$$

$$\#857: \quad x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3}}{B x_n + E x_{n-3}}$$

$$\#858: \quad x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3}}{A + B x_n + E x_{n-3}}$$

$$\#859: \quad x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3}}{A + B x_n + E x_{n-3}}$$

$$\#860: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3}}{B x_n + E x_{n-3}}$$

$$\#861: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3}}{B x_n + E x_{n-3}}$$

$$\#862: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3}}{A + Bx_n + Ex_{n-3}}$$

$$\#863: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3}}{A + Bx_n + Ex_{n-3}}$$

$$\#900: \quad x_{n+1} = \frac{\beta x_n + \epsilon x_{n-3}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#901: \quad x_{n+1} = \frac{\alpha + \beta x_n + \epsilon x_{n-3}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#912: \quad x_{n+1} = \frac{\gamma x_{n-1} + \epsilon x_{n-3}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#913: \quad x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \epsilon x_{n-3}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#916: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \epsilon x_{n-3}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#917: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \epsilon x_{n-3}}{Dx_{n-2} + Ex_{n-3}}$$

$$\#920: \quad x_{n+1} = \frac{\gamma x_{n-1} + \epsilon x_{n-3}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#921: \quad x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \epsilon x_{n-3}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#924: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \epsilon x_{n-3}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#925: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \epsilon x_{n-3}}{Bx_n + Dx_{n-2} + Ex_{n-3}}$$

$$\#932: \quad x_{n+1} = \frac{\beta x_n + \epsilon x_{n-3}}{C x_{n-1} + D x_{n-2} + E x_{n-3}}$$

$$\#933: \quad x_{n+1} = \frac{\alpha + \beta x_n + \epsilon x_{n-3}}{C x_{n-1} + D x_{n-2} + E x_{n-3}}$$

$$\#948: \quad x_{n+1} = \frac{\beta x_n + \gamma x_{n-1} + \epsilon x_{n-3}}{C x_{n-1} + D x_{n-2} + E x_{n-3}}$$

$$\#949: \quad x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \epsilon x_{n-3}}{C x_{n-1} + D x_{n-2} + E x_{n-3}}$$

$$\#984: \quad x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3}}{B x_n + D x_{n-2} + E x_{n-3}}$$

$$\#985: \quad x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3}}{B x_n + D x_{n-2} + E x_{n-3}}$$

$$\#996: \quad x_{n+1} = \frac{\beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{C x_{n-1} + D x_{n-2} + E x_{n-3}}$$

$$\#997: \quad x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2} + \epsilon x_{n-3}}{C x_{n-1} + D x_{n-2} + E x_{n-3}}$$

Open Problem 5 *For each of the above* 101 *fourth-order special cases, establish its boundedness character and investigate the region of global asymptotic stability of its equilibrium point*(*s*).

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