On Periodic Trichotomies

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What is a periodic trichotomy?

3 way split of the qualitative behavior

Let R be a region of our parameters.

Parameters in $R \implies$ ESC $\bar{x}$.

Parameters on $\partial R \implies$ ESC period P.

Parameters outside of $\overline{R} \implies \exists US$.

Example:

$$x_n = \frac{\beta_2 x_{n-2}}{1 + x_{n-1} + x_{n-5}}, \quad n \in \mathbb{N}.$$

$$\beta_2 < 1 \quad \Longrightarrow \quad 0 \ \ is \ \ GAS.$$

$$\beta_2 = 1 \quad \Longrightarrow \quad ESCP_2.$$

$$\beta_2 > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{\beta_3 x_{n-3} + \beta_9 x_{n-9}}{1 + x_{n-1} + x_{n-2} + x_{n-4} + x_{n-5}}, \quad n \in \mathbb{N}.$$

$$\beta_3 + \beta_9 < 1 \quad \Longrightarrow \quad 0 \ \ is \ \ GAS.$$

$$\beta_3 + \beta_9 = 1 \quad \Longrightarrow \quad ESCP_3.$$

$$\beta_3 + \beta_9 > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{\beta_4 x_{n-4} + \beta_6 x_{n-6}}{1 + x_{n-1} + x_{n-5}}, \quad n \in \mathbb{N}.$$

$$\beta_4 + \beta_6 < 1 \quad \Longrightarrow \quad 0 \ \ is \ \ GAS.$$

$$\beta_4 + \beta_6 = 1 \quad \Longrightarrow \quad ESCP_2.$$

$$\beta_4 + \beta_6 > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{\beta_{10} x_{n-10} + \beta_{15} x_{n-15} + \beta_{20} x_{n-20}}{1 + x_{n-1} + x_{n-2} + x_{n-3} + x_{n-4} + x_{n-16}}, \quad n \in \mathbb{N}.$$

$$\beta_{10} + \beta_{15} + \beta_{20} < 1 \quad \Longrightarrow \quad 0 \ \ is \ \ GAS.$$

$$\beta_{10} + \beta_{15} + \beta_{20} = 1 \quad \Longrightarrow \quad ESCP_5.$$

$$\beta_{10} + \beta_{15} + \beta_{20} > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{\beta_{20} x_{n-20} + \beta_{26} x_{n-26}}{1 + x_{n-1} + x_{n-3}}, \quad n \in \mathbb{N}.$$

$$\beta_{20} + \beta_{26} < 1 \quad \Longrightarrow \quad 0 \;\; is \;\; GAS.$$

$$\beta_{20} + \beta_{26} = 1 \quad \Longrightarrow \quad ESCP_2.$$

$$\beta_{20} + \beta_{26} > 1 \quad \Longrightarrow \quad \exists US.$$

We want to find a pattern to these examples.

The delays that are present in the numerator and denominator play an important role.

In order to state a general theorem we need to define the following sets of integers:

$$I_\beta = \{i \in \{1, 2, \ldots, k\} | \beta_i > 0\}.$$

$$I_B = \{j \in \{1, 2, \ldots, k\} | B_j > 0\}.$$

Now we will state a general theorem from:

F.J. Palladino, On periodic trichotomies, *J. Difference Equ. Appl.* **15**(2009), 605-620.

Theorem 1: Consider the k^{th} order rational difference equation,

$$x_n = \frac{\sum_{i=1}^{k} \beta_i x_{n-i}}{1 + \sum_{i=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}.$$

Suppose that the $gcd(I_\beta)$ does not divide j for any $j \in I_B$. Then we have the following:

$$\sum_{i=1}^{k} \beta_i < 1 \quad \Longrightarrow \quad 0 \;\; is \;\; GAS.$$

$$\sum_{i=1}^{k} \beta_i = 1 \quad \Longrightarrow \quad ESCP_{gcd(I_\beta)}.$$

$$\sum_{i=1}^{k} \beta_i > 1 \quad \Longrightarrow \quad \exists US.$$

Example using Theorem 1:

$$x_n = \frac{\beta_{20}x_{n-20} + \beta_{24}x_{n-24}}{1 + x_{n-1} + x_{n-3}}, \quad n \in \mathbb{N}.$$

$$I_\beta = \{20, 24\} \quad and \quad I_B = \{1, 3\}.$$
$$gcd(20, 24) = 4 \quad and \quad 4 \not| 1, 3.$$

So, by Theorem 1:

$$\beta_{20} + \beta_{24} < 1 \quad \Longrightarrow \quad 0 \ \ is \ \ GAS.$$

$$\beta_{20} + \beta_{24} = 1 \quad \Longrightarrow \quad ESCP_4.$$

$$\beta_{20} + \beta_{24} > 1 \quad \Longrightarrow \quad \exists US.$$

Example:
How does adding an α to the numerator change the behavior?

$$x_n = \frac{\alpha + \beta_{20}x_{n-20} + \beta_{24}x_{n-24}}{1 + x_{n-1} + x_{n-3}}, \quad n \in \mathbb{N}.$$

$$I_\beta = \{20, 24\} \quad and \quad I_B = \{1, 3\}.$$
$$gcd(20, 24) = 4 \quad and \quad 4 \not| 1, 3.$$

However we have:

$$\beta_{20} + \beta_{24} < 1 \quad \Longrightarrow \quad \bar{x} \;\; is \;\; GAS.$$

$$\beta_{20} + \beta_{24} = 1 \quad \Longrightarrow \quad ESCP_2.$$

$$\beta_{20} + \beta_{24} > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{\alpha + \beta_{20} x_{n-20} + \beta_{24} x_{n-24}}{1 + x_{n-2} + x_{n-6}}, \quad n \in \mathbb{N}.$$

$$\beta_{20} + \beta_{24} < 1 \quad \Longrightarrow \quad \bar{x} \ \ is \ \ GAS.$$

$$\beta_{20} + \beta_{24} = 1 \quad \Longrightarrow \quad ESCP_4.$$

$$\beta_{20} + \beta_{24} > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{\alpha + \beta_6 x_{n-6} + \beta_{12} x_{n-12}}{1 + x_{n-1} + x_{n-3}}, \quad n \in \mathbb{N}.$$

$$\beta_6 + \beta_{12} < 1 \quad \Longrightarrow \quad \bar{x} \;\; is \;\; GAS.$$

$$\beta_6 + \beta_{12} = 1 \quad \Longrightarrow \quad ESCP_2.$$

$$\beta_6 + \beta_{12} > 1 \quad \Longrightarrow \quad \exists US.$$

Example:

$$x_n = \frac{\alpha + \beta_6 x_{n-6} + \beta_{12} x_{n-12}}{1 + x_{n-3}}, \quad n \in \mathbb{N}.$$

$$\beta_6 + \beta_{12} < 1 \quad \Longrightarrow \quad \bar{x} \;\; is \;\; GAS.$$

$$\beta_6 + \beta_{12} = 1 \quad \Longrightarrow \quad ESCP_6.$$

$$\beta_6 + \beta_{12} > 1 \quad \Longrightarrow \quad \exists US.$$

Now we will state two more general theorems from:

F.J. Palladino, On periodic trichotomies, *J. Difference Equ. Appl.* **15**(2009), 605-620.

Theorem 2: Consider the k^{th} order rational difference equation,

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{1 + \sum_{i=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}.$$

Suppose that the $gcd(I_\beta \cup I_B) = 1$ and i is even for all $i \in I_\beta$, and j is odd for all $j \in I_B$. Then we have the following:

$$\sum_{i=1}^{k} \beta_i < 1 \quad \Longrightarrow \quad \bar{x} \;\; is \;\; GAS.$$

$$\sum_{i=1}^{k} \beta_i = 1 \quad \Longrightarrow \quad ESCP_2.$$

$$\sum_{i=1}^{k} \beta_i > 1 \quad \Longrightarrow \quad \exists US.$$

Theorem 3: Consider the k^{th} order rational difference equation,

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{1 + \sum_{i=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}.$$

Suppose that the $gcd(I_\beta \cup I_B) = \ell$ and $\frac{i}{\ell}$ is even for all $i \in I_\beta$, and $\frac{j}{\ell}$ is odd for all $j \in I_B$. Then we have the following:

$$\sum_{i=1}^{k} \beta_i < 1 \quad \Longrightarrow \quad \bar{x} \;\; is \;\; GAS.$$

$$\sum_{i=1}^{k} \beta_i = 1 \quad \Longrightarrow \quad ESCP_{2\ell}.$$

$$\sum_{i=1}^{k} \beta_i > 1 \quad \Longrightarrow \quad \exists US.$$

Example using Theorem 2:

$$x_n = \frac{\alpha + \beta_6 x_{n-6} + \beta_{12} x_{n-12}}{1 + x_{n-1} + x_{n-3}}, \quad n \in \mathbb{N}.$$

$$gcd(I_\beta \cup I_B) = gcd(\{1,3,6,12\}) = 1.$$

$$I_\beta = \{6, 12\} \quad all \;\; evens.$$

$$I_\beta = \{1, 3\} \quad all \;\; odds.$$

So, Theorem 2 tells us:

$$\beta_6 + \beta_{12} < 1 \quad \Longrightarrow \quad \bar{x} \;\; is \;\; GAS.$$

$$\beta_6 + \beta_{12} = 1 \quad \Longrightarrow \quad ESCP_2.$$

$$\beta_6 + \beta_{12} > 1 \quad \Longrightarrow \quad \exists US.$$

Example using Theorem 3:

$$x_n = \frac{\alpha + \beta_6 x_{n-6} + \beta_{12} x_{n-12}}{1 + x_{n-3}}, \quad n \in \mathbb{N}.$$

$$gcd(I_\beta \cup I_B) = gcd(\{3, 6, 12\}) = 3.$$

$$\left\{\frac{6}{3}, \frac{12}{3}\right\} \quad all \;\; evens.$$

$$\left\{\frac{3}{3}\right\} \quad is \;\; odd.$$

So, Theorem 3 tells us:

$$\beta_6 + \beta_{12} < 1 \quad \Longrightarrow \quad \bar{x} \;\; is \;\; GAS.$$

$$\beta_6 + \beta_{12} = 1 \quad \Longrightarrow \quad ESCP_6.$$

$$\beta_6 + \beta_{12} > 1 \quad \Longrightarrow \quad \exists US.$$

Now we will state two conjectures from:

E. Camouzis and G. Ladas, Three trichotomy conjectures, *J. Difference Equ. Appl.* **8**(2002), 495-500.

Conjecture 1:

Assume that $\alpha \in (0, \infty)$. Then the following period-five trichotomy result is true for the rational equation

$$x_n = \frac{\alpha + x_{n-3}}{x_{n-1}}, n \in \mathbb{N}.$$

$$\alpha > 1 \quad \Longrightarrow \quad \bar{x} \;\; is \;\; GAS.$$

$$\alpha = 1 \quad \Longrightarrow \quad ESCP_5.$$

$$\alpha < 1 \quad \Longrightarrow \quad \exists US.$$

Conjecture 2:

Assume that $\alpha, C \in [0, \infty)$. Then the following period-six trichotomy result is true for the rational equation

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, n \in \mathbb{N}.$$

$$\alpha C^2 > 1 \quad \Longrightarrow \quad \bar{x} \ is \ GAS.$$

$$\alpha C^2 = 1 \quad \Longrightarrow \quad ESCP_6.$$

$$\alpha C^2 < 1 \quad \Longrightarrow \quad \exists US.$$

Directions for further work:

Resolve conjectures 1 and 2.

Find other general families of periodic trichotomies.

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