Improvements to the Spectrum of Known *Z*-cyclic Whist Tournament Designs

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A **whist tournament** on v players, denoted $\mathrm{Wh}(v)$, is a $(v, 4, 3)$ (near) resolvable BIBD. A whist game is a block, (a, b, c, d), of the BIBD and denotes that the partnership $\{a, c\}$ opposes the partnership $\{b, d\}$. The design is subject to the (whist) conditions that every player is a partner of every other player exactly once and is an opponent of every other player exactly twice.

A whist design is said to be Z-**cyclic** if the players are elements in $Z_m \cup \mathcal{A}$ where $m = v$, $\mathcal{A} = \emptyset$ when $v \equiv 14$ and $m = v - 1$, $\mathcal{A} = \{\infty\}$ when $v \equiv 04$. It is also required that the rounds be cyclic. That is to say, the rounds can be labelled, $R_1, R_2, \ldots$, in such a way that R_{j+1} is obtained by adding $+1 \pmod{m}$ to every element in R_j.

Example - Wh(5)

The initial round of a Z-cyclic Wh(5) is given by the single game $(1, 3, 4, 2)$. The remaining 4 rounds are obtained by developing this initial round modulo 5.

Example - Wh(9)

Let $X = \{0, 1, 2, \ldots, 8\}$. A Wh$(|X|)$ is given by the following 9 rounds:

$(8, 2, 4, 6)$ $(1, 3, 5, 7)$
$(4, 7, 0, 5)$ $(2, 6, 3, 1)$
$(0, 1, 8, 3)$ $(7, 5, 6, 2)$
$(2, 3, 7, 4)$ $(5, 8, 0, 6)$
$(7, 6, 1, 0)$ $(3, 4, 8, 5)$
$(1, 5, 2, 8)$ $(6, 0, 4, 3)$
$(3, 8, 6, 7)$ $(0, 2, 1, 4)$
$(6, 4, 5, 1)$ $(8, 7, 2, 0)$
$(5, 0, 3, 2)$ $(4, 1, 7, 8)$

The focus here is on $\mathrm{Wh}(v)$ with $v \equiv 1 \mod 4$. For such designs there is one player that sits out in each round. We will find it convenient to represent such whist designs in a matrix format as follows.

Definition

Suppose there exists a $\mathrm{Wh}(v)$, $v \equiv 1 \mod 4$. Define a v by v matrix, $\mathcal{W}h(v)$, to be such that the i-th row of $\mathcal{W}h(v)$ is the i-th round of the $\mathrm{Wh}(v)$ arranged so that $\mathcal{W}h(v)_{i,1}$ is the player that sits out in round i and $\mathcal{W}h(v)_{i,4s-2}$, $\mathcal{W}h(v)_{i,4s-1}$, $\mathcal{W}h(v)_{i,4s}$, $\mathcal{W}h(v)_{i,4s+1}$ are the players, in the order given, in the s-th game of the i-th round, $s = 1, 2, \ldots, (v-1)/4$.

In general, there is no order attached to the games in any round of a $\mathrm{Wh}(v)$. Thus the matrix given in the following example is one of many that could be constructed.

Example We illustrate this definition for the Wh(9) given above.

$$\mathcal{W}h(9) = \begin{pmatrix} 0 & 8 & 2 & 4 & 6 & 1 & 3 & 5 & 7 \\ 8 & 4 & 7 & 0 & 5 & 2 & 6 & 3 & 1 \\ 4 & 0 & 1 & 8 & 3 & 7 & 5 & 6 & 2 \\ 1 & 2 & 3 & 7 & 4 & 5 & 8 & 0 & 6 \\ 2 & 7 & 6 & 1 & 0 & 3 & 4 & 8 & 5 \\ 7 & 1 & 5 & 2 & 8 & 6 & 0 & 4 & 3 \\ 5 & 3 & 8 & 6 & 7 & 0 & 2 & 1 & 4 \\ 3 & 6 & 4 & 5 & 1 & 8 & 7 & 2 & 0 \\ 6 & 5 & 0 & 3 & 2 & 4 & 1 & 7 & 8 \end{pmatrix}$$

Definition A **pairwise balanced design**, $\mathrm{PBD}(v, K, \lambda)$ is a collection of subsets (blocks) of a v-set S such that (i) the size of each block is in K and is less than v and (ii) each pair of elements of S occur together in exactly λ of the blocks. **Example** A $\mathrm{PBD}(21, \{5\}, 1)$ is obtained by developing the block $\{3, 6, 7, 12, 14\}$ modulo 21. Note that there are 21 blocks each of size 5.

Theorem A Let G be a group such that $|G| = 4n + 1$. Let $B_1, \ldots, B_t$ be subsets of G such that $|B_k| \equiv 1 \mod 4$ for all k and the collection $\{g \cdot B_k : g \in G, k \in \{1, \ldots, t\}\}$ forms a PBD$(4n + 1, K, 1)$ with $K = \{|B_k| : k \in \{1, \ldots, t\}\}$ and $g \cdot B_k = \{g \cdot b : b \in B_k\}$. Then there is a Wh$(|G|)$, whose players are elements in G, with the property that there exists a round, R_e, of the Wh$(|G|)$ for which the collection $\{g \cdot R_e : g \in G\}$ provides all rounds of the Wh$(|G|)$. That is to say, there exists a G-cyclic Wh$(|G|)$.

Proof The existence of a Wh$(|G|)$ is guaranteed by Theorem 11.2.1 in Ian Anderson's book "Combinatorial Designs and Tournaments". We proceed, now, to demonstrate that the G-cyclic nature of the PBD$(4n+1, K, 1)$ leads to the fact that this Wh$(|G|)$ is G-cyclic . For each block B_k construct a Wh$(|B_k|)$ whose players are the elements of the block B_k and construct the corresponding matrix $\mathcal{W}h(|B_k|) = (w_{ij}^k)$. Define $g \cdot \mathcal{W}h(|B_k|) = (g \cdot w_{ij}^k) = (w_{ij}^{k'})$. Since, for each $g \in G$, "left multiplication" by g is a bijection from B_k to $g \cdot B_k$ it is clear that $g \cdot \mathcal{W}h(|B_k|)$ over $g \cdot B_k$ is merely a relabeling of the $Wh(|B_k|)$ over B_k.

Hence $g \cdot \mathcal{W}h(|B_k|)$ is a representation of a whist design whose players are the elements of $g \cdot B_k$. Consider the collection of blocks $\{g \cdot B_k : g \in G, k \in \{1, 2, \ldots, t\}\}$ and the collection of whist designs $\{g \cdot \mathcal{W}h(|B_k|) : g \in G, k \in \{1, 2, \ldots, t\}\}$. For each $g \in G$ a round, R_g, of the Wh$(|G|)$ is obtained as the union of the rounds in the $g \cdot \mathcal{W}h(|B_k|)$'s in which g sits out. (As in the constructive proof of Theorem 11.2.1 in Ian Anderson's book.) Thus the matrix $\mathcal{W}h(|G|)$ is a composite formed from the $g \cdot \mathcal{W}h(|B_k|)$'s. Consider the round R_{e_G} where e_G is the identity element in G. Observe that e_G sits out in round s of $g_1 \cdot \mathcal{W}h(B_k) \Leftrightarrow g_1 \cdot w^k_{s,1} = e_G \Leftrightarrow g \cdot g_1 \cdot w^k_{s,1} = g \Leftrightarrow g$ sits out in round s of $g \cdot g_1 \cdot Wh(B_k)$. Thus we have shown that (a, b, c, d) is a game in the round that e_G sits out $\Leftrightarrow (g \cdot a, g \cdot b, g \cdot c, g \cdot d)$ is a game in the round that g sits out. We conclude that R_{e_G} serves as an initial round of a G -cyclic Wh$(|G|)$. **QED**

The following three designs were built using the Z-cyclic PBD$(v, \{5, 9\}, 1)$, $v = 133, 153, 213$ found in the paper of R. J. R. Abel and F. E. Bennett "Quintessential PBDs and PBDs with prime power block sizes $\geq$ 8" and the construction embodied in the proof of Theorem A. The examples of Wh(153) and Wh(213) are new Z-cyclic whist designs. Although a Z-cyclic Wh(133) is known, our example of Wh(133) is a new result in that this design has both the directedwhist and orderedwhist properties. We have also constructed a Z-cyclic TWh(1057) which is a new result. This TWh(1057) was built using the theorem of Buratti and Zuanni, the Singer difference set for the projective plane of order 32 and a TWh(33).

Z-**cyclic DOWh**(133)

$(123, 3, 39, 61)$, $(1, 7, 47, 103)$,
$(2, 6, 102, 38)$, $(46, 122, 132, 60)$,
$(100, 58, 131, 130)$, $(4, 36, 120, 44)$,
$(54, 32, 40, 127)$, $(116, 96, 129, 126)$,
$(94, 95, 84, 101)$, $(64, 8, 22, 97)$,
$(93, 76, 14, 56)$, $(86, 89, 87, 125)$,
$(119, 72, 79, 75)$, $(111, 73, 42, 62)$,
$(31, 77, 33, 20)$, $(91, 30, 69, 37)$,
$(49, 113, 10, 57)$, $(13, 71, 17, 11)$,
$(24, 112, 53, 5)$, $(48, 128, 107, 19)$,
$(88, 114, 109, 29)$, $(80, 104, 85, 59)$,
$(26, 74, 45, 21)$, $(27, 108, 92, 9)$,
$(83, 124, 99, 18)$, $(81, 115, 106, 65)$,
$(41, 68, 50, 16)$, $(34, 117, 52, 25)$,
$(35, 78, 63, 12)$, $(51, 121, 66, 23)$,
$(43, 110, 98, 28)$, $(70, 105, 82, 15)$,
$(67, 118, 90, 55)$.

Z-**cyclic DOWh**(153)

$(142, 5, 35, 82)$, $(4, 17, 44, 89)$,
$(1, 13, 85, 31)$, $(40, 138, 149, 78)$,
$(84, 77, 152, 148)$, $(12, 30, 137, 39)$,
$(65, 18, 27, 140)$, $(125, 72, 141, 136)$,
$(118, 122, 107, 135)$, $(54, 9, 47, 123)$,
$(126, 98, 38, 45)$, $(109, 114, 113, 144)$,
$(115, 71, 88, 76)$, $(106, 75, 7, 60)$,
$(68, 108, 69, 53)$, $(146, 64, 99, 81)$,
$(46, 100, 11, 55)$, $(16, 93, 28, 15)$,
$(2, 58, 121, 25)$, $(23, 119, 151, 56)$,
$(33, 128, 130, 96)$, $(63, 97, 120, 95)$,
$(32, 57, 90, 34)$, $(3, 94, 104, 24)$,
$(21, 101, 150, 91)$, $(70, 129, 132, 80)$,
$(10, 62, 83, 59)$, $(49, 73, 143, 52)$,
$(6, 48, 67, 26)$, $(20, 61, 147, 42)$,
$(22, 127, 133, 41)$, $(19, 111, 131, 105)$,
$(86, 112, 134, 92)$, $(8, 51, 87, 37)$,
$(29, 79, 145, 43)$, $(14, 116, 124, 50)$,
$(36, 110, 139, 102)$, $(66, 103, 117, 74)$.

Z-**cyclic Wh**(213)

$(175, 5, 26, 81)$,	$(4, 16, 41, 104)$,	$(47, 92, 162, 61)$,
$(1, 12, 100, 22)$,	$(37, 171, 209, 77)$,	$(51, 112, 143, 98)$,
$(99, 76, 212, 208)$,	$(11, 21, 170, 36)$,	$(70, 168, 182, 121)$,
$(65, 10, 25, 201)$,	$(159, 88, 202, 197)$,	$(31, 152, 199, 101)$,
$(187, 191, 149, 203)$,	$(78, 15, 55, 192)$,	$(14, 115, 166, 45)$,
$(188, 134, 40, 63)$,	$(172, 177, 176, 198)$,	$(39, 68, 155, 48)$,
$(173, 132, 148, 137)$,	$(158, 136, 23, 94)$,	$(58, 106, 126, 97)$,
$(113, 150, 114, 71)$,	$(190, 109, 135, 125)$,	$(87, 184, 193, 145)$,
$(64, 142, 38, 79)$,	$(43, 119, 54, 42)$,	$(20, 165, 204, 107)$,
$(2, 95, 181, 35)$,	$(33, 179, 211, 93)$,	$(9, 116, 174, 29)$,
$(60, 178, 180, 146)$,	$(86, 120, 153, 118)$,	$(103, 130, 160, 111)$,
$(32, 67, 127, 34)$,	$(3, 75, 131, 62)$,	$(53, 164, 183, 156)$,
$(59, 128, 210, 72)$,	$(13, 151, 154, 69)$,	$(30, 186, 194, 83)$,
$(56, 141, 200, 138)$,	$(82, 144, 157, 85)$,	$(19, 102, 205, 49)$,
$(6, 139, 167, 50)$,	$(44, 161, 207, 133)$,	$(8, 57, 110, 27)$,
$(89, 163, 169, 117)$,	$(28, 80, 124, 74)$,	$(66, 90, 195, 73)$,
$(46, 96, 185, 52)$,	$(7, 129, 147, 24)$,	$(18, 91, 108, 84)$,
$(17, 140, 206, 122)$,	$(105, 189, 196, 123)$.	

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