A Counterexample to the Approximation Problem in Banach Spaces

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A Banach space B is said to have the approximation property (a.p. for short) if every compact operator from a Banach space into B can be approximated in the norm topology for operators by finite rank operators.

The classical approximation problem is the question whether all Banach spaces have a.p. In (Enflo, 1973) a negative answer to this question is given by constructing a Banach space which does not have the a.p.

We will present the first 3 lemmas from this paper. We will also discuss very briefly the strategy for the remainder of this paper.

Before discussing the paper, (Enflo,1973), we should fist give some background and motivation for the problem.

Let B denote a Banach space over $\mathbb{R}$(or $\mathbb{C}$). A Schauder basis is a sequence $\{b_n\}_{n=1}^{\infty}$ of elements of B such that for every element $v \in B$ there exists a unique sequence $\{\alpha_n\}_{n=1}^{\infty}$ of elements of $\mathbb{R}$(or $\mathbb{C}$) so that

$$v = \sum_{n=1}^{\infty} \alpha_n b_n.$$

Here convergence is with respect to the norm topology.

An example.

Consider a Hilbert space H. We know (from Functional analysis part 1) that H admits an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. This basis is a Schauder basis with $\alpha_n = \langle v, e_n \rangle$.

A less trivial example.

Consider ℓ^p with $1 \leq p < \infty$. Recall that for $v \in \ell^p$ we may write $v = (v_1, v_2, \cdots, v_n, \cdots)$. Choose $b_n = e_n$ and $\alpha_n = v_n$ and we have a Schauder basis for ℓ^p, $1 \leq p < \infty$.
Proof.

Consider that

$$\left\| v - \sum_{n=1}^{M} v_n e_n \right\|_p = ||(0, 0, \cdots 0, v_{M+1}, v_{M+2}, \cdots)||_p = \left(\sum_{n=M+1}^{\infty} |v_n|^p \right)^{\frac{1}{p}}.$$

Also

$$\left(\sum_{n=1}^{M} |v_n|^p \right) + \left(\sum_{n=M+1}^{\infty} |v_n|^p \right) = \left(\sum_{n=1}^{\infty} |v_n|^p \right).$$

If viewed as sequences indexed by M, the three sequences above are all bounded monotone sequences of real numbers hence convergent.

So we get

$$\lim_{M\to\infty}\left(\sum_{n=1}^{M}|v_n|^p\right)+\lim_{M\to\infty}\left(\sum_{n=M+1}^{\infty}|v_n|^p\right)=\lim_{M\to\infty}\left(\sum_{n=1}^{\infty}|v_n|^p\right).$$

So

$$\left(\sum_{n=1}^{\infty}|v_n|^p\right)+\lim_{M\to\infty}\left(\sum_{n=M+1}^{\infty}|v_n|^p\right)=\left(\sum_{n=1}^{\infty}|v_n|^p\right).$$

Thus

$$\lim_{M\to\infty}\left(\sum_{n=M+1}^{\infty}|v_n|^p\right)=0.$$

So

$$\lim_{M\to\infty}\left\|v-\sum_{n=1}^{M}v_ne_n\right\|_p=0.$$

More examples.

Most Banach spaces that you know and love.

In fact the classical basis problem posed by Banach asks whether every separable Banach space has a Schauder basis. This problem went unsolved for decades, despite garnering significant interest from eminent mathematicians.
The Scottish Café was the cafe in Lwów where mathematicians from the Lwów school of Mathematics met and spent their afternoons discussing mathematical problems. Difficult problems were gathered into a book, which came to be called the "Scottish Book". Prizes were awarded for solution of any of the problems in the book. For one notoriously difficult problem, Mazur promised a live goose as a reward. It was later shown that Mazur's "Goose Problem" was equivalent to Banach's basis problem.

A Banach space is said to have the bounded approximation property (b.a.p. for short) if there is a net (S_n) of finite rank operators on B such that $S_n \to I$ in strong operator topology and such that there is a uniform bound on the norms of the S_n:s. It was proved by Grothendiek that the b.a.p. implies the a.p. and that for reflexive Banach spaces the b.a.p. is equivalent to the a.p. (see Grothendiek, 1955).

So what is actually done by Enflo in (Enflo,1973) is to construct a separable reflexive Banach space which fails to have the b.a.p.

Notes before we begin.

If $T_n v \to Tv$ in norm for all $v \in B$ then we say $T_n \to T$ in the strong operator topology.

A Banach space over $\mathbb{R}$(or $\mathbb{C}$) is reflexive if it is isomorphic to its own double dual. (Isometry by the Hahn-Banach theorem with pointwise evaluation operator.)

From pointset Topology a Banach space is said to be separable if it has a countable dense subset.

Lemma 0.(not proved in Enflo,1973)

If B has a Schauder basis then B has the b.a.p.
Proof.
Take S_m to be the projection operator onto the first m terms of b_n.
Then by the definition of a Schauder basis

$$S_m(v) = \sum_{n=1}^{m} \alpha_n b_n \to \sum_{n=1}^{\infty} \alpha_n b_n = v = I(v).$$

Now suppose that the basis problem was answered in the affirmative. Then every separable Banach space would have a Schauder basis and hence the b.a.p.

Enflo constructs a separable reflexive Banach space which fails to have the b.a.p., providing a counterexample to all three problems:
The basis problem,
the “Goose problem”,
and the approximation problem.

Theorem 1. (Enflo, 1973)

There exists a separable reflexive Banach space, B, with a sequence (M_n) of finite-dimensional subspaces, $dim(M_n) \to \infty$ when $n \to \infty$, and a constant C such that for every T of finite rank

$$||T - I||_{\overline{M_n}} \geq 1 - \frac{C||T||}{log(dim(M_n))}.$$

In particular, B does not have the approximation property and B does not have a Schauder basis.

We shall say that an operator T on B is a finite expansion operator on B if for every k

$$Te_k = \sum_i a_{ik} e_i,$$

where the sum is finite.

Lemma 1. (Enflo, 1973)
Let B be a Banach space generated by a sequence of vectors $\{e_j\}_{j=1}^{\infty}$ which is linearly independent (for finite sums). If T is a continuous finite rank operator on B, then, for every $\epsilon > 0$ there is a finite rank finite expansion operator T_1 on B so that

$$||T - T_1|| < \epsilon.$$

Proof.
Put $||T|| = K$. Assume that $f_1, \ldots, f_r$ is a basis for the range of T. Approximate $f_1, \ldots, f_r$ by $\hat{f}_1, \ldots, \hat{f}_r$ all of finite expansion in $\{e_j\}_{j=1}^{\infty}$ so that for real numbers $b_1, \ldots b_r$ we have

$$\left|\left|\sum_{j=1}^{r} b_j f_j - \sum_{j=1}^{r} b_j \hat{f}_j\right|\right| \leq \frac{\epsilon || \sum_{j=1}^{r} b_j f_j ||}{K}.$$

Now if $T(x) = \sum_j b_j f_j$ then put $T_1(x) = \sum_j b_j \hat{f}_j$. This gives

$$||T(x) - T_1(x)|| = \left\| \sum_{j=1}^{r} b_j f_j - \sum_{j=1}^{r} b_j \hat{f}_j \right\| \leq \frac{\epsilon || \sum_{j=1}^{r} b_j f_j ||}{K} \leq \epsilon ||x||.$$

Let B be a Banach space generated by a sequence of vectors $\{e_j\}_{j=1}^{\infty}$ which is linearly independent (for finite sums) and let T be a finite expansion operator on B. Let M be a finite subset of the $e_j : s$. We will put

$$Tr(M,T) = \sum_{e_i \in M} a_{ii} \quad and \quad \overline{Tr}(M,T) = \frac{1}{|M|} \sum_{e_i \in M} a_{ii},$$

where $|M|$ is the cardinality of M.

Let $\{e_j\}_{j=1}^{\infty}$ be a sequence of non-zero vectors which generate a Banach space. We shall say tnhat $\{e_j\}_{j=1}^{\infty}$ has property A if for each finite sum $\sum_{j=1}^{r} a_j e_j$ we have $||\sum_{j=1}^{r} a_j e_j|| \geq ||a_k e_k||$ for all k with $1 \leq k \leq r$. If M is a set of vectors in a Banach space we will denote by $\overline{M}$ the closed subspace of B generated by M.

Lemma 2. (Enflo, 1973)
Let B be a Banach space generated by a sequence of vectors $\{e_j\}_{j=1}^{\infty}$ which has property A. Let T be a finite expansion operator on B and let M be a finite subset of $\{e_j\}_{j=1}^{\infty}$. Then

$$|\overline{Tr}(M,T)| \leq ||T||_{\overline{M}}.$$

Proof.
If $e_k \in M$ we have

$$|a_{kk}| \leq \frac{||a_{kk}e_k||}{||e_k||} \leq \frac{||\sum_i a_{ik}e_i||}{||e_k||} = \frac{||T(e_k)||}{||e_k||} \leq ||T||_{\overline{M}}.$$

Lemma 3. (Enflo, 1973)
Let B be a Banach space generated by a sequence of vectors $\{e_j\}_{j=1}^{\infty}$ which has property A. Assume that there is a sequence M_m of mutually disjoint finite subsets of $\{e_j\}_{j=1}^{\infty}$ and constants $a > 1$ and $K > 0$ such that

(i) $dim(M_{m+1}) \geq (dim(M_m))^a, \quad m \in \mathbb{N},$

(ii) $|\overline{Tr}(M_{m+1}, \hat{T}) - \overline{Tr}(M_m, \hat{T})| \leq \frac{K||\hat{T}||}{log(dim(M_m))}, \quad m \in \mathbb{N},$

for every finite expansion operator $\hat{T}$ on B.
Then there is a constant C such that for every finite rank operator T on B

$$||T - I||_{\overline{M_m}} \geq 1 - \frac{C||T||}{log(dim(M_m))}.$$

Proof.

By Lemma 1 it suffices to prove the result for finite rank finite expansion operators on B. Since T has finite rank we have $\overline{Tr}(M_k, T) \to 0$ when $k \to \infty$. Lemma 2 and the assumptions of Lemma 3 then give

$$||I - T||_{\overline{M_m}} \geq |\overline{Tr}(M_m, I - T)| \geq 1 - \sum_{k=m}^{\infty} |\overline{Tr}(M_{k+1}, T) - \overline{Tr}(M_k, T)|$$

$$\geq 1 - K||T|| \sum_{k=m}^{\infty} \frac{1}{log(dim(M_k))} \geq 1 - \frac{C||T||}{log(dim(M_m))},$$

with $C = \frac{K}{1-a^{-1}}$.

References

1. P. Enflo, A counterexample to the approximation property in Banach spaces. Acta Math. 130, 309b•“317(1973).
2. A. Grothendieck, Produits tensoriels topologiques et espaces nucleaires. Memo. Amer. Math. Soc. 16 (1955).
3. The Scottish Book.

Lemma 1. (More details not found in Enflo, 1973)
Let B be a Banach space generated by a sequence of vectors $\{e_j\}_{j=1}^{\infty}$ which is linearly independent (for finite sums). If T is a continuous finite rank operator on B, then, for every $\epsilon > 0$ there is a finite rank finite expansion operator T_1 on B so that

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Proof.
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$$\left|\left|\sum_{j=1}^{r} b_j f_j - \sum_{j=1}^{r} b_j \hat{f}_j\right|\right| \leq \frac{\epsilon||\sum_{j=1}^{r} b_j f_j||}{K}.$$

Choose

$$||f_i - \hat{f}_i|| < \delta.$$

Where

$$\delta \leq \frac{\epsilon||\sum_{j=1}^{r} b_j f_j||}{K\sum_{j=1}^{r} |b_j|}.$$